Quantum Dynamics & Topological Phases

electronic and photonic semiconductors

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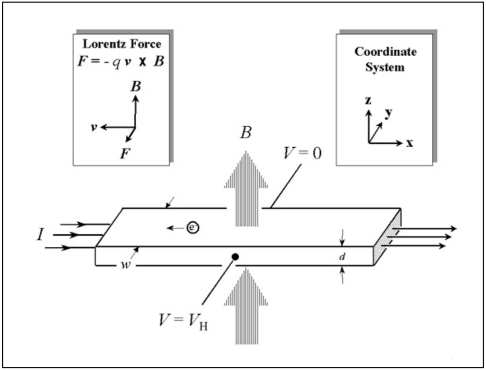
Outline

- 1879 - E. H. Hall inferred from the Maxwell's equations the existence of *transverse currents* (classical Hall effect).
- 1980 - K. von Klitzing observed the quantization of the transverse conductance at $T \sim 0$ K^o (quantum Hall effect).
- 1981 - First theoretical explanation by *R. B. Laughlin* (flux tube argument). Topological Quantum Numbers (TQN) appear on the scene.
- Nowadays quantum Hall systems provide the prototypical (hence simplest) example of Topological Insulator (TI).

Experimental setting for the QHE

Gas of 2-dimensional independent-magnetic-Bloch-electrons, $\mathbb{Z}^2$ crystal lattice, B uniform orthogonal magnetic field.

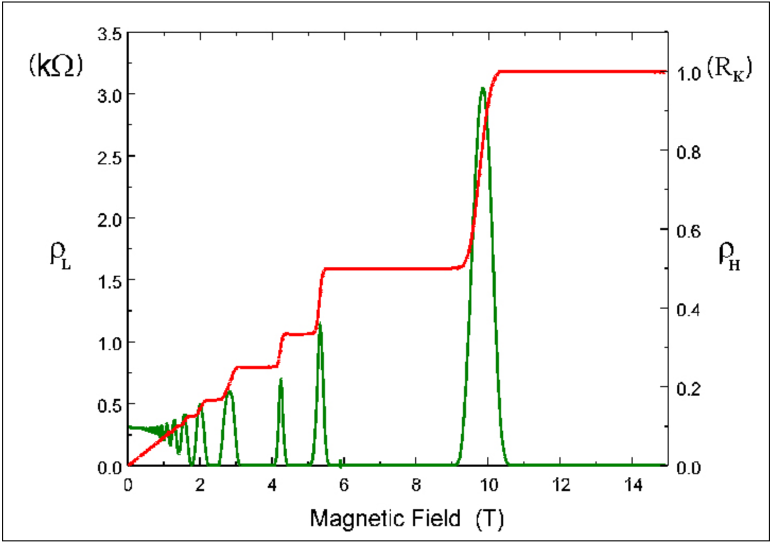


Dimensionless parameter: $h_B := \frac{\Phi_0}{B}$, $\Phi_0 := \frac{hc}{e}$ (magnetic flux quantum).

Hofstadter regime: $h_B \gg 1$ ($B \to 0$, usual experimental setting).

Harper regime: $h_B \ll 1$ ($B \to \infty$, optical lattices).

Quantization of the resistivity (GaAs-GaAlAs heterojunction)



Hall resistance and conductance: $\rho_H := {\sigma_H}^{-1} = \frac{1}{n} R_H$.
von Klitzing constant $R_H := \frac{h}{e^2}$. $n = 1, 2, 3, 4, 6, 8, \ldots$

Outline

The pioneering and seminal work (*TKNN-paper*)

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

PACS numbers: 72.15.Gd, 72.20. Mg, 73.90.+b

paved, for the first time, the way to explain the QHE by TQN:

gapped electronic systems $\Leftrightarrow$ topology of vector bundles

- The paper is a collection of very interesting ideas.
- The ideas are developed without mathematical rigor.
- The results are generalizable to a wide range of situations.

From Symmetries to Bloch-bundle ...

$\mathcal{H}$ = separable Hilbert space,
$U : \mathbb{Z}^N \to \mathscr{U}(\mathcal{H})$ = unit. rep. (*wandering syst. + algebraically. comp.*),
$H \in \mathscr{L}(\mathcal{H})$, $\quad H = H^\dagger \quad$ (*not necessarily bounded*),

Definition

H has a $\mathbb{Z}^N$-symmetry iff $\quad [f(H); U(n)] = 0, \qquad \forall\, n \in \mathbb{Z}^N, \ \forall\, f \in L^\infty(\mathbb{R}).$

Theorem (G. D. & G. Panati: Spectral Days, Santiago, 2012)

Assume that H has a $\mathbb{Z}^N$-symmetry, then:

(i) *a Bloch-Floquet (type) unitary decomposition exists:*

$$\mathcal{H} \longrightarrow \int_{\mathbb{T}^N}^{\oplus} \mathrm{d}k\ \mathcal{H}_k, \qquad\qquad f(H) \longrightarrow \int_{\mathbb{T}^N}^{\oplus} \mathrm{d}k\ f(H)_k;$$

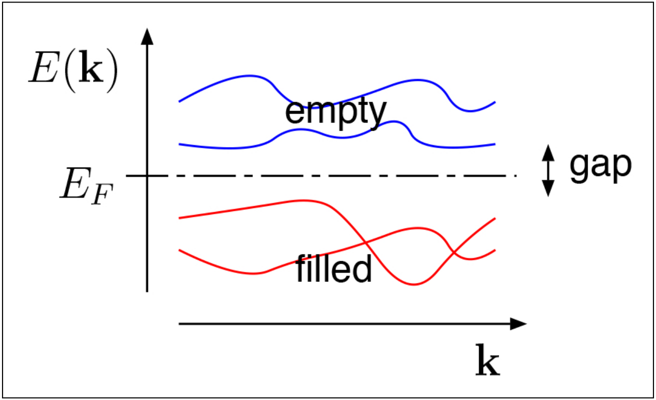
(ii) *if $P \in C(\sigma(H))$ such that $P(H) = P(H)^2$ (gap condition) then:*

$$\pi : \mathscr{E}(P) \longrightarrow \mathbb{T}^N, \qquad\qquad \mathscr{E}(P) = \bigcup_{k \in \mathbb{T}^N} [P(H)_k\, \mathcal{H}_k]$$

is a Hermitian vector (Hilbert) bundle which is uniquely determined.

... more in general: Band spectrum

$$H(k)\ \psi_j(k) \;=\; E_j(k)\ \psi_j(k)\ , \qquad\qquad k \in \mathbb{B}$$



☞ Usually an energy gap separates the filled valence bands from the empty conduction bands. The Fermi level E_F characterizes the gap.

Gap condition and Fermi projection

- An isolated family of energy bands is any (finite) collection $\{E_{j_1}(\cdot), \ldots, E_{j_m}(\cdot)\}$ of energy bands such that

$$\min_{k\in\mathbb{B}} \operatorname{dist}\left(\bigcup_{s=1}^{m}\{E_{j_s}(k)\}\;,\;\bigcup_{j\in\mathcal{I}\setminus\{j_1,\ldots,j_m\}}\{E_j(k)\}\right) \;=\; C_g \;>0\;.$$

This is usually called “gap condition”.

- An isolated family is described by the “Fermi projection”

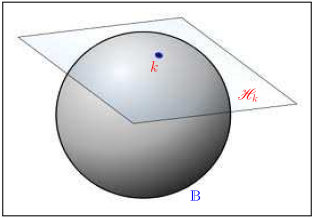
$$P_F(k) \;:=\; \sum_{s=1}^{m}\,|\psi_{j_s}(k)\rangle\langle\psi_{j_s}(k)|\;.$$

This is a continuous projection-valued map

$$\mathbb{B}\;\ni\;k\;\longmapsto\;P_F(k)\;\in\;\mathcal{K}(\mathcal{H})\;.$$

- For each $k \in \mathbb{B}$
$$\mathcal{H}_k \ := \ \mathrm{Ran}\ P_F(k) \ \subset \ \mathcal{H}$$
is a subspace of $\mathcal{H}$ of fixed dimension m.

- The collection
$$\mathcal{E}_F \ := \ \bigsqcup_{k \in \mathbb{B}} \mathcal{H}_k$$
is a topological space (said total space) and the map
$$\pi : \ \mathcal{E}_F \ \longrightarrow \ \mathbb{B}$$
defined by $\pi(k, \mathrm{v}) = k$ is continuous (and open).



This is a complex vector bundle (of rank m) called “Bloch-bundle”.

Definition (Topological phases)

Let $\mathbb{B} \ni k \longmapsto H(k)$ be a TQS with an isolated family of m energy bands and associated Bloch bundle $\mathscr{E}_F \longrightarrow \mathbb{B}$. The topological phase of the system is specified by

$$[\mathscr{E}_F] \in \mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) .$$

- $\mathrm{M} \cap \Phi$ Dictionary -

- **"Ordinary" quantum system:**

TQS in a trivial phase

$\Updownarrow$

Trivial vector bundle, $0 \equiv [\mathbb{B} \times \mathbb{C}^m]$

$\Updownarrow$

Exists a global frame of continuous Bloch functions

- **Allowed (adiabatic) deformations:**

Transformations which doesn't alter the nature of the system

$\Updownarrow$

Vector bundle isomorphism

$\Updownarrow$

Stability of the topological phase

Theorem (Peterson, 1959)

If $\dim(\mathbb{B}) \leqslant 4$ *then*

$$\begin{aligned} \mathrm{Vec}^1_{\mathbb{C}}(\mathbb{B}) &\simeq H^2(\mathbb{B}, \mathbb{Z}) \\ \mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) &\simeq H^2(\mathbb{B}, \mathbb{Z}) \oplus H^4(\mathbb{B}, \mathbb{Z}) \qquad (m \geqslant 2) \end{aligned}$$

and the isomorphism

$$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) \ni [\mathscr{E}] \longmapsto (c_1, c_2) \in H^2(\mathbb{B}, \mathbb{Z}) \oplus H^4(\mathbb{B}, \mathbb{Z})$$

is given by the first two Chern classes (notice $c_2 = 0$ *if* $m = 1$*).*

☞ $\mathbb{B}$ a connected orientable closed manifold of dimension 2 (*e.g.* $\mathbb{T}^2$, $\mathbb{S}^2$, ...). Then $H^2(\mathbb{B}, \mathbb{Z}) = \mathbb{Z}$ and the integro-differential expression holds

$$\boxed{c_1(\mathscr{E}_F) \equiv \frac{\mathrm{i}}{2\pi} \int_{\mathbb{B}} \mathrm{Tr}\Big(P_F(k) \big[\partial_{k_1} P_F(k), \partial_{k_2} P_F(k) \big] \Big) \, \mathrm{d}k}$$

Linear response theory [Bellissard, Schulz-Baldes, Rebolledo, ...]

- $H = H^*$ self-adjoint element of a C^*-algebra $\mathcal{A}$.
- Liouvillian evolution $\frac{\mathrm{d}}{\mathrm{d}t} A = \mathcal{L}_H(A)$, with $\mathcal{L}_H(A) := \mathrm{i}\,[A, H]$, $A \in \mathcal{A}$.
- $\rho = \rho(t = 0)$ is given by $f_{\beta,\mu}(H) \in \mathcal{A}$ (Fermi-Dirac distribution).
- $\mathcal{A}$ has a gradient $\nabla = (\nabla_1, \ldots, \nabla_d)$, *e.g.* $\nabla_j(A) = -\mathrm{i}\,[X_j, A]$.
- $\mathcal{A}$ has an integral $\mathcal{T}$, *e.g.* trace per unit volume (thermodynamic limit).

Definition

The averaged current of an observable $\mathfrak{O}$ on the initial state ρ drifted by the perturbation $\mathfrak{P}$ is given by

$$J_{\mathfrak{O},\mathfrak{P}}(\lambda) \; := \; \lim_{\delta \to 0^+} \int_0^{+\infty} \mathrm{e}^{-\delta t}\, \mathcal{T}\Big(\rho_{\mathfrak{P}}(t)\, \mathfrak{O}\Big) \,, \qquad \lambda \ll 1$$

where $\rho_{\mathfrak{P}}(t) := \mathrm{e}^{t\mathcal{L}_{H+\lambda\,\mathfrak{P}}}(\rho)$ is the full perturbed evolution.

$$J_{\mathfrak{O},\mathfrak{P}}(\lambda) \; = \; \lambda\, \sigma_{\mathfrak{O},\mathfrak{P}}(\beta, \mu) \; + \; \mathcal{O}(\lambda^2)$$

defines the Kubo coefficient.

- Let $\mathcal{A}$ be the C^*-algebra of periodic or random operators in $\dim = 2$.
- $\mathfrak{O} = \nabla_1(H) := -\mathrm{i}\,[X_1, H]$ and $\mathfrak{P} = X_2$.
- $f_{\beta,\mu}$ the Fermi-Dirac distribution with μ in a gap (periodic case) or in a localization region (random case) of H.

In the limit of zero temperature $\beta = +\infty$ the Kubo coefficient is

$$\sigma_{1,2}(+\infty, \mu) \;=\; \frac{\mathrm{i}}{2\pi}\mathcal{T}\Big(P_F\,[\nabla_1(P_F), \nabla_2(P_F)]\Big)$$

$P_F = \lim_{\beta\to+\infty} f_{\beta,\mu}(H)$ is the Fermi projection at energy $E_F = \mu$.

Theorem (Kubo-Chern duality)

$$\sigma_{1,2}(+\infty, \mu = E_F) \;=\; c_1(\mathscr{E}_F)$$

$\swarrow$ *functional analysis*
operator algebras

$\searrow$ *geometry & topology*
of vector bundles

Bloch-Landau Hamiltonian

Densely defined on $L^2(\mathbb{R}^2)$ by

$$H_{\mathrm{BL}} := \frac{\hbar^2}{2m}\left[\left(-i\frac{\partial}{\partial x} - \frac{\pi}{h_B}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{\pi}{h_B}x\right)^2\right] + V_{\mathrm{per}}(x,y)$$

where V_{per} is a $\mathbb{Z}^2$-periodic (crystal) potential, $h_B \propto \frac{1}{B}$.

Theorem (D. 2010)

There are semiclassical adiabatic reductions

$H_{Hof} := \wp(U_0, V_0)$ *on* $L^2(\mathbb{T}^2)$

$H_{BL} \xrightarrow{B \to 0} H_{Hof}$

$H_{Hof} \overset{\text{isospectrality}}{\longleftrightarrow} H_{Har}$ (*isospectrality*)

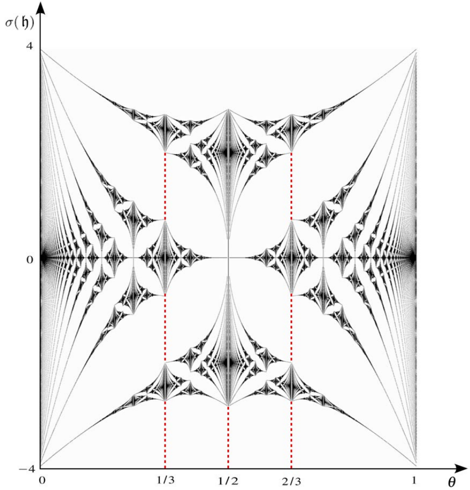
$H_{BL} \xrightarrow{B \to \infty} H_{Har}$

$H_{Har} := \wp(U_\infty, V_\infty)$ *on* $L^2(\mathbb{R})$

which are (asymptotic) unitary equivalence.

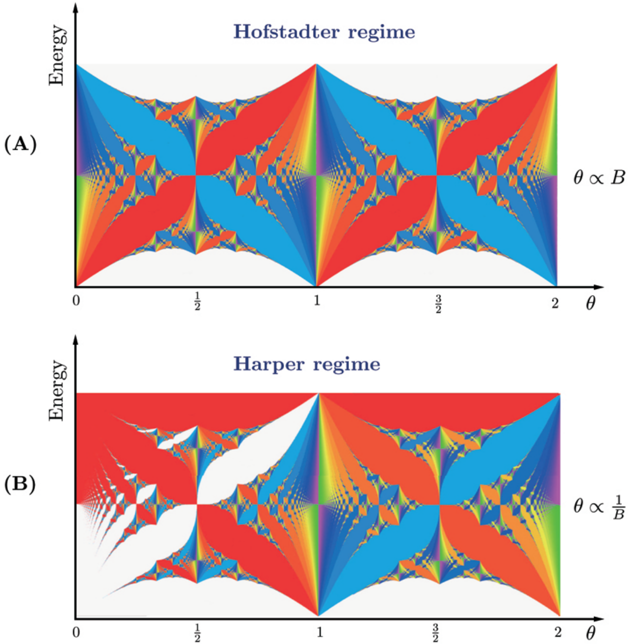
Simplest (formal) model (universal Hofstadter operator)

$$\wp(\mathfrak{u},\mathfrak{v}) = \mathfrak{u}+\mathfrak{u}^{-1}+\mathfrak{v}+\mathfrak{v}^{-1} =: \mathfrak{h}\,, \qquad \in C^*\Big(\mathfrak{u},\mathfrak{v}\;\big|\;\mathfrak{u}\mathfrak{v} = e^{i\theta}\mathfrak{v}\mathfrak{u}\Big)\,.$$

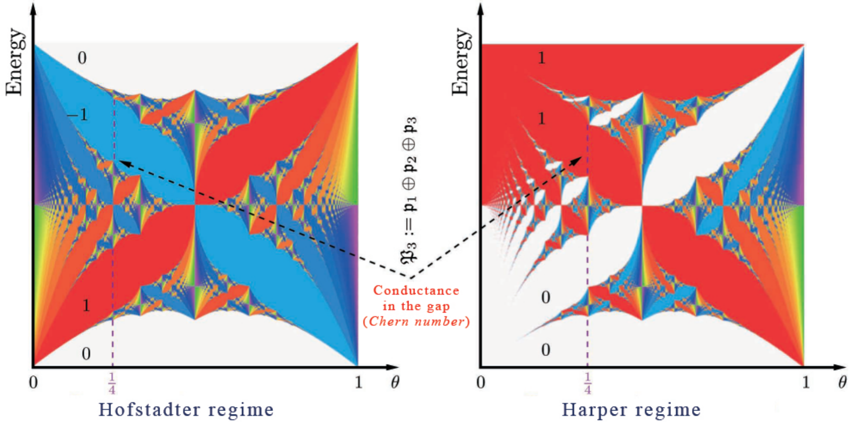


$$\sigma(H_{\mathrm{Hof}}^{B_1}) = \sigma(\mathfrak{h}) = \sigma(H_{\mathrm{Har}}^{B_2}) \qquad \theta = B_1 = B_2^{-1}$$

Color-coded quantum butterflies (courtesy of J. Avron)



Beyond isospectrality ... topology can distinguish the models



Color = Hall conductance = Chern number.

$$\theta = M/N,\ \ M = 1,\ \ N = 4,\ \ j = 3,\ \ \ \mathfrak{P}_3 = \mathfrak{p}_1 \oplus (\mathfrak{p}_2 \oplus \mathfrak{p}_3)$$

$$4\ \underbrace{C_{\mathrm{Har}}(\mathfrak{P}_3)}_{=\,1} + 1\ \underbrace{C_{\mathrm{Hof}}(\mathfrak{P}_3)}_{=\,-1} = 3 \qquad \text{(diophantine TKNN-equation)}$$

1. Quantum Hall effect, Hofstadter's butterfly, TKNN duality
   - Phenomenology of the QHE
   - Bloch-bundle: physical meaning and topology
   - From the Kubo's formula to Chern numbers
   - Adiabatic reduction and butterflies
   - The TKNN-equation as a geometric duality
2. Topological insulators, CAZ classification, twisted bundles
   - Periodic table of topological phases
   - Spectral flow and Index theory
   - Classification principle in 2D
3. Maxwell dynamics and the topological phases of the light
   - Photonic crystals
   - Photonic topological insulators
   - A lot of work to do ...

Theorem (D. 2010)

Let $\theta = \frac{M}{N}$ and $\mathfrak{p}$ a gap-projection of $\mathfrak{h}$. Let $\mathscr{E}^0_{\mathfrak{p}}$ (resp. $\mathscr{E}^\infty_{\mathfrak{p}}$) be the Bloch-bundle associated with $\mathfrak{p}$ in the Hofstadter (resp. Harper) limit and $C_{\mathrm{Hof}}(\mathfrak{p}) := c_1(\mathscr{E}^0_{\mathfrak{p}})$ (resp. $C_{\mathrm{Har}}(\mathfrak{p}) := c_1(\mathscr{E}^\infty_{\mathfrak{p}})$) the related Chern number. Then

$$N\ C_{\mathrm{Har}}(\mathfrak{p})\ +\ M\ C_{\mathrm{Hof}}(\mathfrak{p})\ =\ \mathcal{T}(\mathfrak{p})\ .$$

If $\mathfrak{P}_k := \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_k$ is the total projection up to the k-th gap

$$N\ C_{\mathrm{Har}}(\mathfrak{P}_k)\ +\ M\ C_{\mathrm{Hof}}(\mathfrak{P}_k)\ =\ k \qquad \textit{(TKNN-equation)}\ .$$

☞ The proof is a consequence of the more fundamental geometric duality

$$\boxed{\alpha^*\left(\mathscr{E}^\infty_{\mathfrak{p}}\right)\ \simeq\ \beta^*\left(\mathscr{E}^0_{\mathfrak{p}}\right)\ \otimes\ \det\left(\mathscr{E}^\infty_{\mathbb{1}}\right) \qquad \alpha, \beta \in C(\mathbb{T}^2)}$$

☞ Improved in [G. D., G. Landi. Adv. Theor. Math. Phys. **16** (2012)] .

Fundamental Symmetries

Let H be a Hamiltonian (self-adjoint operator) on a complex Hilbert space endowed with a (anti-linear) complex conjugation C.

Time Reversal Symmetry (TR)

H has a TR-sym. if there exists a unitary operator T such that:

$$C\,H\,C \;=\; +\,T\,H\,T^{*}\,, \qquad \begin{cases} \text{even} & \text{if} \quad CTC = +T^{*} \quad i.e.\ (CT)^2 = +\mathbb{1} \\ \text{odd} & \text{if} \quad CTC = -T^{*} \quad i.e.\ (CT)^2 = -\mathbb{1}\,. \end{cases}$$

Particle-Hole Symmetry (PH)

H has a PH-sym. if there exists a unitary operator I such that:

$$C\,H\,C \;=\; -\,I\,H\,I^{*}\,, \qquad \begin{cases} \text{even} & \text{if} \quad CIC = +I^{*} \quad i.e.\ (CI)^2 = +\mathbb{1} \\ \text{odd} & \text{if} \quad CIC = -I^{*} \quad i.e.\ (CI)^2 = -\mathbb{1}\,. \end{cases}$$

Chiral Symmetry ($\mathcal{X}$)

H has a $\mathcal{X}$-sym. if there exists a unitary operator $\mathcal{X}$ such that:

$$H \;=\; -\,\mathcal{X}\,H\,\mathcal{X}^{*}\,, \qquad \mathcal{X}^2 = \pm\mathbb{1} \qquad (e.g.\ \ \mathcal{X}' = \mathrm{i}\,\mathcal{X})\,.$$

Cartan-Altland-Zirnbauer (CAZ) classification

CAZ	*TR*	*PH*	$\mathcal{X}$	$d=1$	$d=2$	$d=3$	$d=4$	Physics
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	QHE
AI	$+$	0	0	0	0	0	$2\mathbb{Z}$	TR-invariant
AII	$-$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	systems
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	chiral
BDI	$+$	$(+)$	1	$\mathbb{Z}$	0	0	0	systems
CII	$-$	$(-)$	1	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
D	0	$+$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	BdG
C	0	$-$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	systems
DIII	$-$	$+$	(1)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	(supercond.)
CI	$+$	$-$	(1)	0	0	$2\mathbb{Z}$	0	

☞ **Remark:** Classification for free fermions not for Bloch electrons !!

Open questions:

(1) Structural analysis of the geometry underlying different classes;

(2) Definition of the topological objects which provide the classification;

(3) Association between topological invariants and physical observables;

(4) Bulk-edge correspondence in each of the 10 classes;

(5) Extension to the random case (stability).

Possible approaches ... and some result:

- (1) and (2) done for classes AI and AII by looking at cohomology:

  [G. D., K. Gomi. J. Geom. Phys. **86**, (2014)]
  [G. D., K. Gomi. Submitted to Comm. Math. Phys., (2014)]

- A different approach to (2) and (4) is based on index theory:

  [G. D., H. Schulz-Baldes. Canad. Math. Bull., (2014)]
  [G. D., H. Schulz-Baldes. Ann. H. Poincaré, (2014)]

- (3) some results for BdG classes: spin and thermal QHE:

  [G. D., H. Schulz-Baldes. In preparation]

- (5) stability for BdG classes (localization á la Aizenman-Molchanov):

  [G. D., M. Drabkin, H. Schulz-Baldes. Pastur fest 75th birthday]

Outline

Classification via the Noether-Fredholm index

Harper-like Hamiltonian with random perturbation on $\ell^2(\mathbb{Z}^2)$

$$H_\omega = \wp\left(S_1^B, S_2^B\right) + \lambda V_\omega$$

where $\wp$ is a polynomial, $S_j^B := \mathrm{e}^{(-1)^j B X_{j+1}} S_j$ are the magnetic translations, $\lambda \in \mathbb{R}$, $\omega \in \Omega$ randomness.

Theorem (Connes, Bellissard, Kunz, Avron, Seiler, Simon ...)

Let $P_F := \chi_{(-\infty, E_F]}(H)$ be the Fermi projection with E_F in a gap or in a regime of Anderson localization. Then

$$P_F\, U\, P_F \quad \text{is Fredholm}, \qquad U := \frac{X_1 + \mathrm{i}\, X_2}{|X_1 + \mathrm{i}\, X_2|}$$

and, defined $\mathrm{Ind}(A) = \dim\ker(A) - \dim\ker(A^*)$, one has

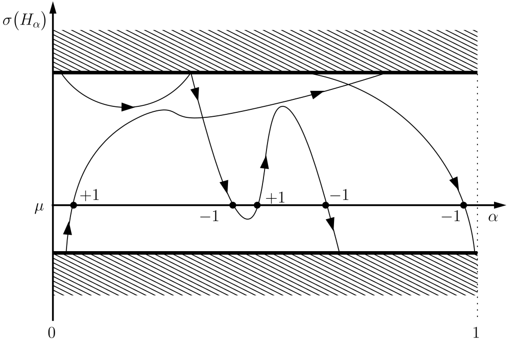
$$\mathrm{Ind}\,(P_F\, U\, P_F) = \frac{1}{\mathrm{i}\, 2\pi} \mathcal{T}\Big(P_F\, [[X_1, P_F], [X_2, P_F]]\Big) = c_1(\mathscr{E}_F)\,.$$

Laughlin argument (1981) as a Spectral Flow

- Let us insert a magnetic flux tube $\alpha \in [0,1]$ through the unit cell $[0,1]^2 \subset \mathbb{Z}^2$.
- The magnetic translations change as $S_j^B \mapsto S_j^{\alpha,B} := \mathrm{e}^{\,\mathrm{i}\,\alpha\, f_j(X_1,X_2)} S_j^B$.
- $H_\alpha - H_{\alpha=0}$ is compact (only discrete spectrum moves).

Theorem (G. D., H. Schulz-Baldes. Ann. H. Poincaré, (2014))

$$\text{Spectral Flow}\Big([0,1] \ni \alpha \mapsto H_\alpha \quad \textit{through} \quad \mu = E_F\Big) \;=\; c_1(\mathscr{E}_F)\,.$$



Outline

Odd symmetric Fredholm operators and $\mathbb{Z}_2$-index

- Let $\mathcal{H}$ be a separable Hilbert space with complex conjugation C.
- T is a unitary anti-involution $T^* = T^{-1} = -T$ and real $CTC = T$.
- A is odd-symmetric $\Leftrightarrow$ $T\,\overline{A}\,T^* = A^*$ where $\overline{A} := C\,A\,C$.

Theorem (G. D., H. Schulz-Baldes. Canad. Math. Bull., (2014))

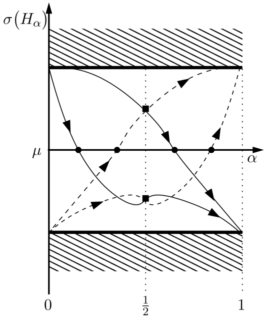
(i) *The space* $\mathbb{F}_{\rm os}(\mathcal{H}) := \{\text{odd} - \text{symmetric Fredholm operators}\}$ *has two connected components classified by the* $\mathbb{Z}_2$*-index*

$$\mathrm{Ind}_{\mathbb{Z}_2}(A) \;:=\; (-1)^{\dim\,\mathrm{Ker}(A)}\;.$$

(ii) *If H is a self-adjoint Hamiltonian with odd TRS and $P_F := \chi_{(-\infty,E_F]}(H)$ with E_F in a gap or in a regime of Anderson localization, then* $P_F\;U\;P_F \in \mathbb{F}_{\rm os}(\mathcal{H})$ *and*

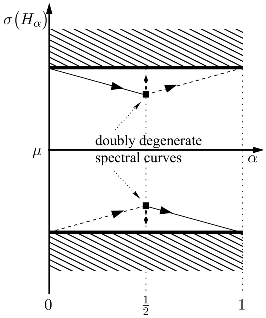
$$\mathrm{Ind}_{\mathbb{Z}_2}\,(P_F\;U\;P_F) \;=\; \mathrm{Spectral\ Flow}\Big([0,1/2]\ni\alpha\mapsto H_\alpha\Big)/\mathrm{mod}.2$$

Even half−flux

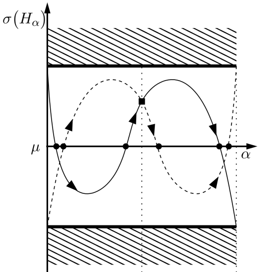


⇒
homotopic
deformation

Standard trivial diagram

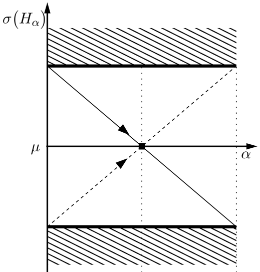


Odd half−flux



⇒
homotopic
deformation

Standard non−trivial diagram



Theorem (G. D., H. Schulz-Baldes. Ann. H. Poincaré, (2014))

- Classification principle in 2D -

All the topological CAZ phases for 2D tight-binding systems can be describe by the unique Fredholm operator $P_F\ U\ P_F$.

- For classes AI, AIII, BDI, CI and CII only trivial phase.
- For classes A and D one has a $\mathbb{Z}$-classification given by

$$c_1(\mathscr{E}_F) = \mathrm{Ind}\,(P_F\ U\ P_F)\ .$$

- For class C the symmetry implies that $\mathrm{Ind}\,(P_F\ U\ P_F) \in 2\mathbb{Z}$.
- For classes AII and DIII the symmetry implies the vanishing of the primary invariant $\mathrm{Ind}\,(P_F\ U\ P_F) = 0$. The secondary invariant $\mathrm{Ind}_{\mathbb{Z}_2}\,(P_F\ U\ P_F)$ is well defined and provides the $\mathbb{Z}_2$-classification.

Open questions:

(1) Extension to higher dimensions;

(2) Extension to the continuous case;

(3) Description of the weak invariants.

Outline

Maxwell's equations

$$\epsilon\, \partial_t \boldsymbol{E} = +\nabla \times \boldsymbol{H} \qquad \mu\, \partial_t \boldsymbol{H} = -\nabla \times \boldsymbol{E} \qquad \text{(dynamic)}$$

$$\nabla \cdot \epsilon\, \boldsymbol{E} = 0 \qquad \nabla \cdot \mu\, \boldsymbol{H} = 0 \qquad \text{(no sources)}$$

Conditions on the material weights

$$W := \begin{pmatrix} \epsilon^{-1} & \chi \\ \chi^* & \mu^{-1} \end{pmatrix} \Rightarrow \begin{cases} (1) & 0 < c_1 \mathbb{1} \leqslant W^{-1}, W \leqslant c_2 \mathbb{1} \\ (2) & W = W^* \quad \text{(lossless)} \\ (3) & W \text{ is frequency-independent} \end{cases}$$

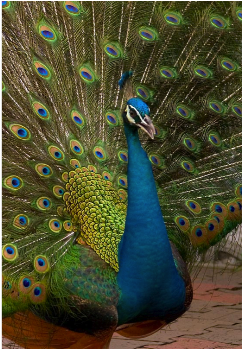
Schrödinger-type light-dynamics

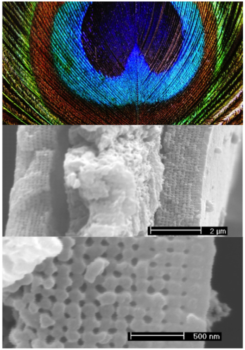
$$\mathrm{i}\, \partial_t \underbrace{\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}}_{\Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6)} = M \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}, \qquad M := \underbrace{\begin{pmatrix} \epsilon^{-1} & \chi \\ \chi^* & \mu^{-1} \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} 0 & +\mathrm{i}\, \nabla^{\times} \\ -\mathrm{i}\, \nabla^{\times} & 0 \end{pmatrix}}_{\mathrm{Rot}}$$

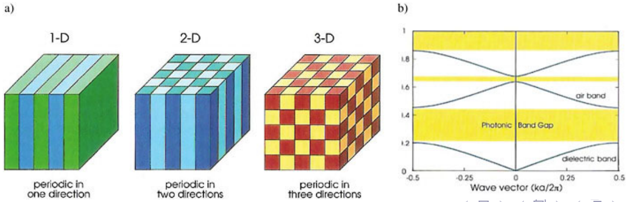
$$\|\Psi\|_W^2 := \langle \Psi, W^{-1}\, \Psi \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^6)} = 2\, \mathscr{E}(\boldsymbol{E}, \boldsymbol{H}), \quad \text{energy density of the e.m. field.}$$

"A photonic crystal is to light what a crystalline solid is to an electron"

ϵ, μ and χ are $\mathbb{Z}^3$-periodic (usually $\chi = 0$) $\Rightarrow$ photonic band gap







Outline

Symmetries of the Maxwell operator

- In the vacuum ($W = \mathbb{1}$)

$$M_{\mathrm{vac}} = \mathrm{Rot} = \sigma_2 \otimes \nabla^{\times}$$

hence $T_j := \sigma_j \otimes \mathbb{1}$ and $J_j := C\, T_j$ ($j = 1, 2, 3$) are symmetries.

- In a PhC ($W \neq \mathbb{1}$) the symmetries depends on the weights ϵ, μ, χ.

Theorem (G. D., M. Lein. *Ann. Phys.* **350**, (2014))

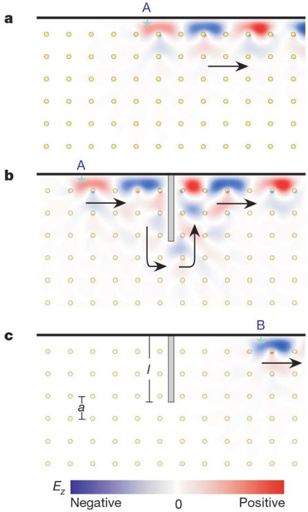
- Exhaustive CAZ classification for PhC's -

9 of 10 of the CAZ classes can be theoretically realized with PhC's. 6 have already been realized in experiments.

Symmetries present	CAZ class	Reduced K-group in dimension			
		$d = 1$	$d = 2$	$d = 3$	$d = 4$
None	A	0	$\mathbb{Z}$	$(\mathbb{Z}^3)$	$\mathbb{Z} \oplus (\mathbb{Z}^6)$
$J \equiv +\mathrm{TR}$	AI	0	0	0	$\mathbb{Z}$
$T \equiv \chi$	AIII	$\mathbb{Z}$	$(\mathbb{Z}^2)$	$\mathbb{Z} \oplus (\mathbb{Z}^3)$	$(\mathbb{Z}^8)$
$C \equiv +\mathrm{PH}$	D	$\mathbb{Z}_2$	$(\mathbb{Z}_2^2) \oplus \mathbb{Z}$	$(\mathbb{Z}_2^3 \oplus \mathbb{Z}^3)$	$(\mathbb{Z}_2^4 \oplus \mathbb{Z}^6)$
$T \equiv \chi, C \equiv +\mathrm{PH}$	BDI	$\mathbb{Z}$	$(\mathbb{Z}^2)$	$(\mathbb{Z}^3)$	$(\mathbb{Z}^4)$
$J_2 \equiv -\mathrm{PH}, J_3 \equiv +\mathrm{TR}$	CI	0	0	$\mathbb{Z}$	$(\mathbb{Z}^4)$

Photonic protected phases

Photonic phases protected by back-scattering have been recently observed:



[Z. Wang, Y. D. Chong, J. D. Joannopoulos, M. Soljačić. Nature **461**, (2009)]

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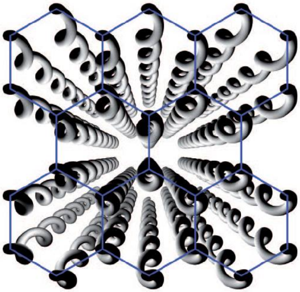

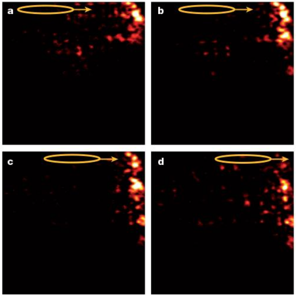

[M. C. Rechtsman *et al.*. Nature **496**, (2013)]

Open questions:

(1) A complete first-principles theory able to explain topological phases;

(2) Topological phases an physical observables (Kubo-Chern formula, ...);

(3) Stability under disorder (random operators and n.c.-geometry).

First results: semiclassical behavior

Ingredients for a space-adiabatic reduction:

(1) A distinction between fast and slow degrees of freedom;

(2) A dimensionless adiabatic parameter e $\lambda \ll 1$ which quantifies the separation of scales between the crystal and the external perturbation;

(3) A relevant gapped part of the spectrum Σ for the unperturbed dynamics.

Theorem (G. D., M. Lein. *Commun. Math. Phys.* **332**, (2014))

(i) *There is a super-adiabatic projection Π_λ associated to Σ such that*

$$[M_\lambda, \Pi_\lambda] = \mathscr{O}(\lambda^\infty) ;$$

(ii) *The full dynamic is adiabatically approximated*

$$\left\| \left(\mathrm{e}^{-\mathrm{i}\, t M_\lambda} - \mathrm{e}^{-\mathrm{i}\, t \mathrm{Op}(\mathscr{M}_{\mathrm{eff}})} \right) \Pi_\lambda \right\| = \mathscr{O}\big((1 + |t|)\lambda^\infty\big)$$

(iii) *The symbol $\mathscr{M}_{\mathrm{eff}}$ describes the semiclassical dynamics and*

$$\mathscr{M}_{\mathrm{eff}} = \mathscr{M}_{\mathrm{eff},0} + \underbrace{\lambda \mathscr{M}_{\mathrm{eff},1}}_{\text{Berry phase + Poynting tensor}} + \mathscr{O}(\lambda^2)$$

Thank you for your attention