Numerical and analytical aspects of contact Hamiltonian systems

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Outline

- Motivations and Examples

PART One: Background and Overview

- Contact Hamiltonian system
- Numerical Approaches

PART Two:

- More background
- Symmetries characterization

Why contact?

Symplectic manifold is the natural arena for conservative physics; i.e. the value of the function that geometrically induces the dynamics, the **Hamiltonian** function, does not change during the motion.

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Symplectic manifold is the natural arena for conservative physics; i.e. the value of the function that geometrically induces the dynamics, the **Hamiltonian** function, does not change during the motion. What happens when the energy is not conserved?

For example when dissipation is present: **viscous drag**, **electrical circuit**, **thermodynamics**, ...

Part 1: Background and Overview

Definition (Contact Manifold)

A couple (M, Δ), *where:*

- M *is an odd-dimensional manifold,*
- $\Delta \subset TM$ *is a distribution of codimension* 1, *maximally non-integrable,* $[\Delta, \Delta] \not\subseteq \Delta$

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Proposition (Reeb vector field)

On (M,η) *there exist an unique vector field* R *that satisfies:* $\eta(R) = 1$ *and* $d\eta(R, \cdot) = 0$.

Contact Transformations

Contact Transformations

Definition

A diffeomorphism $\psi : (M, \eta) \to (N, \alpha)$ *is a contactomorphism or a contact transformation if*

$$\psi^* \alpha = f \eta, \qquad f : M \to \mathbb{R}_0$$

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Furthermore, if $f := 1$*, the diffeomorphism* ψ *is called exact contactomorphism.*

Definition

An infinitesimal contactomorphism on (M, η) *is a vector field* W *such that:*

$$L_W \eta = g_W \eta, \quad g_W : M \to \mathbb{R}.$$

If $g_W := 0$*,* W *is called strict infinitesimal contactomorphism.*

Contact Hamiltonian systems

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Remark:

$$L_{X_{\mathcal{H}}}\eta = d\left(-\mathcal{H}\right) + \left(d\mathcal{H} - R(\mathcal{H})\eta\right) = -R(\mathcal{H})\eta.$$

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Hamiltonian vector fields are contactomorphism.

Darboux Theorem in contact (Hamiltonian systems)

In Darboux coordinates $x = (Q_i, P_i, S)$ where the contact form takes the form:

$$\eta = dS - P_i dQ_i \qquad d\eta = dQ_i \wedge dP_i$$

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the equations of motion are:

$$\begin{cases} \dot{Q}_i = \dfrac{\partial \mathcal{H}}{\partial P_i} \\ \dot{P}_i = -\dfrac{\partial \mathcal{H}}{\partial Q_i} - P_i \dfrac{\partial \mathcal{H}}{\partial S} \\ \dot{S}_i = P_i \dfrac{\partial \mathcal{H}}{\partial P_i} - \mathcal{H} \end{cases}$$

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Proposition

The evolution of the Hamiltonian function under its own flow is given by:

$$\dot{\mathcal{H}} = -\mathcal{H} R(\mathcal{H}).$$

Proposition

The commutation of two Hamiltonian vector fields on the same contact manifold (M, η) *is again a contact Hamiltonian vector field.*

Some useful results

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Proposition

The composition of Hamiltonian flows is again a Hamiltonian flow.

Jacobi Brackets,

On a contact manifold (M, η) is naturally endowed with a **Jacobi structure** (local Lie algebra in Kirillov sense). Therefore, η induces a map:

$$\{\cdot,\cdot\}_\eta : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

that is **bi-linear**, and satisfies the **Jacobi identity** but it does not satisfy the Leibniz rule. The definition depends on the Reeb vector field and the skew-symmetric vector field $\Lambda(\cdot,\cdot)$.

Proposition

The Jacobi brackets of two Hamiltonian functions:

$$[X_f, X_g] = -X_{\{f,g\}_\eta}$$

Jacobi Brackets, ... in coordinates

Jacobi brackets can be expressed in the following way:

$$\{f,g\}_\eta = X_g f + f\ R(g) = -X_f g - g\ R(f).$$

that means in coordinates:

$$\{f,g\}_{ds-pdq} = \left(f\frac{\partial g}{\partial s} - g\frac{\partial f}{\partial s}\right) + p\left(\frac{\partial f}{\partial s}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial s}\right) + \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right).$$

Jacobi Brackets, and conserved quantities

Definition

Let $(M, \eta, \mathcal{H})$ *a contact Hamiltonian system. If* $f : M \to \mathbb{R}$ *has a vanishing Jacobi bracket with* $\mathcal{H}$*, i.e.*

$$\{f, \mathcal{H}\}_\eta = 0$$

then we say that f *is in involution with* $\mathcal{H}$*.*

Proposition

If two functions f, g *on a contact manifold* (M, η) *commute their evolution satisfies:*

$$X_f g = -gR(f).$$

Their ratio f/g *is conserved, but also any function of degree* 0 *in* f *and* g *is conserved.*

Numerical integrators

- **Splitting Numerical Integrators**

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  We can construct by the composition:

$$S_2(\tau) : M \to M$$
$$x_0 \mapsto \circ_{i=1}^{n-1} \left(\Phi^{\frac{\tau}{2}}_{h_{n-i}} \right) \circ \Phi^{\tau}_{n} \circ \left(\circ_{i=1}^{n} \Phi^{\frac{\tau}{2}}_{h_i} \right) (x_0)$$

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\end{aligned}
$$

  $\Rightarrow$[Bravetti-Seri-Vermeeren-Z 2020] the composition of the flows $\Phi^t_{h_i}$ is a **contactomorphism**, so there exist a **modified Hamiltonian** $\tilde{h}_\tau$

$$|\mathcal{H}(Q,P,S) - \tilde{h}_\tau(Q,P,S)| \sim O(\tau^2).$$

Numerical integrators

- **Splitting Numerical Integrators**
- **Lagrangian Numerical integrators** The variational integrator relies on the **Herglotz variational principle**:

$$S(t) = \int_0^t \underbrace{P \frac{\partial \mathcal{H}}{\partial P} - \mathcal{H}(Q, P, S)}_{\mathcal{L}(Q,P,S)} \, dt$$

and the discretization of the Lagrangian.

Numerical integrators

- **Splitting Numerical Integrators**
- **Lagrangian Numerical integrators**
- **Neural Network** It learns a parametric function:

$$\Theta : M \times \mathbb{R}^n \to \mathbb{R},$$

that is equivalent to fixing n parameters to approximate some chosen trajectories.

Neural Network

The aim is to learn the Hamiltonian function from trajectories in the phase space $(q_i(t), p_i(t), s(t))$. So we want to learn a parametric map (the set of parameters $\{\theta_i\}$ is fixed):

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which minimizes the following loss function:

$$L(Q, P, S) := \|\dot{x} - X_{\Theta_{\{\theta_i\}}}\|^2(Q, P, S)$$

where $\dot{x}(Q, P, S)$ is the "velocity" field on the contact phase space, and $X_{\Theta_{\{\theta_i\}}}$ is the Hamiltonian vector field induced by $\Theta_{\{\theta_i\}}(Q, P, S)$, that is

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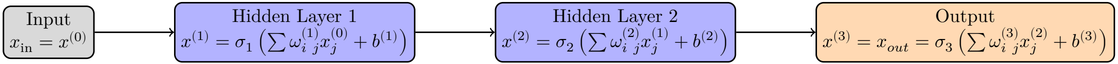
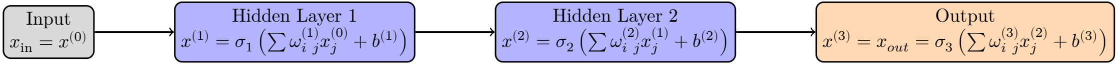
$$X_\Theta = \frac{\partial \Theta}{\partial P}\frac{\partial}{\partial Q} - \left(\frac{\partial \Theta}{\partial Q} + P\frac{\partial \Theta}{\partial S}\right)\frac{\partial}{\partial P} + \left(P\frac{\partial \Theta}{\partial P} - \Theta\right)\frac{\partial}{\partial S}$$



- The set of parameters $\{\theta_i\}$ is divided into groups ω_i that correspond to the layers.

Neural Network



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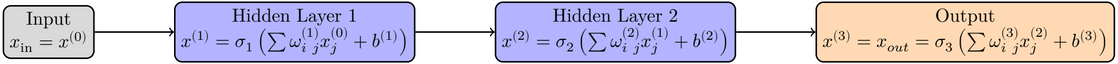

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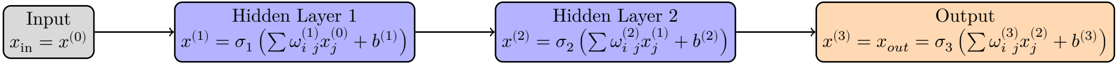

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- The training data set is generated by two different approaches: IVP-integrator (Runge-Kutta) 4(5) and contact splitting integrators.
- The training of the neural network is performed through an Adam-Optimizer.

Neural Network: Results

$$\mathcal{H} = \frac{P^2}{2} + \frac{Q^2}{2} + \gamma S.$$

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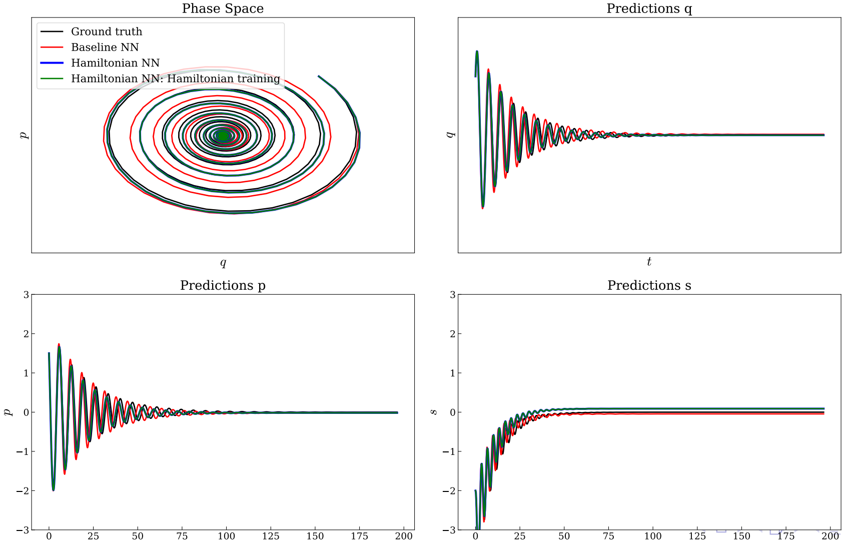

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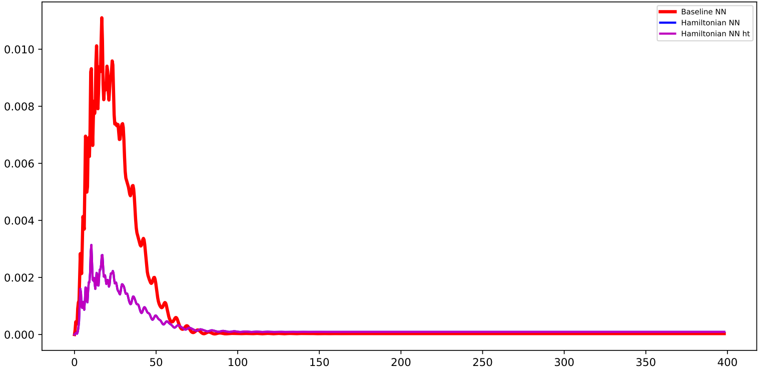


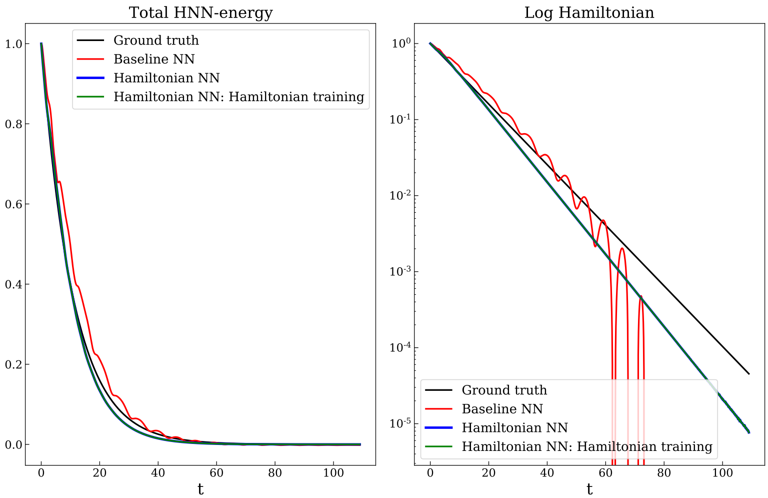
Neural Network: Results

$$\dot{\mathcal{H}} = -\frac{\partial \mathcal{H}}{\partial S}\mathcal{H}$$

so

$$\mathcal{H}(t) = H_0 e^{-\gamma t} \Rightarrow \log(\mathcal{H})(t) \sim -\gamma t$$

Neural Network: Results



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$$\begin{cases} \dot{q} = p \\ \dot{p} = -q - ps \\ \dot{s} = \dfrac{p^2}{2} - \dfrac{q^2}{2} - \dfrac{s^2}{2} + 1 \end{cases}$$

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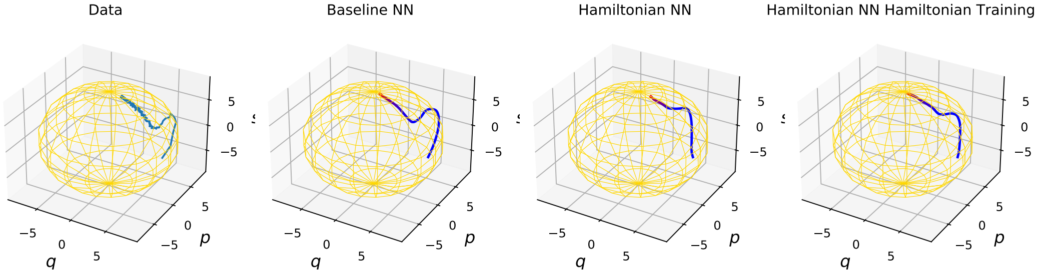

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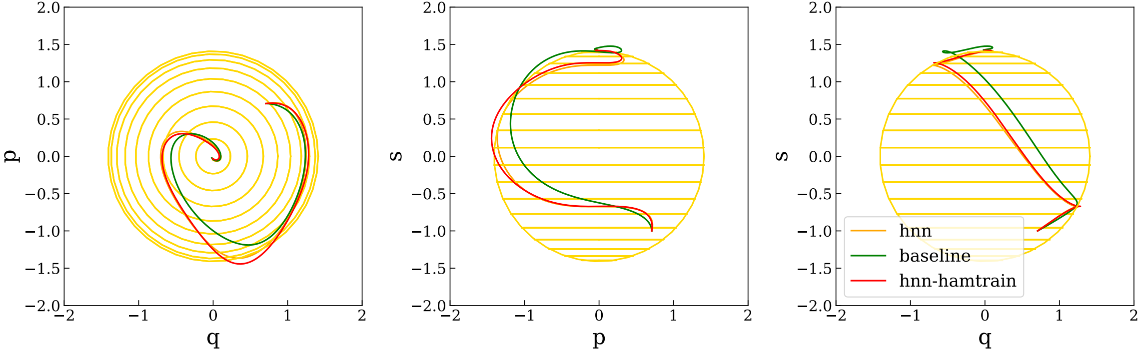

Part 2: Symmetries

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for two functions $a, g \in C^\infty(M)$.

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for two functions $a, g \in C^{\infty}(M)$.
Dynamical similarity[9] is a vector field $W \in \mathfrak{X}(M)$ such that

$$[W, X_{\mathcal{H}}] = \phi_W \, X_{\mathcal{H}},$$

for a smooth function ϕ_W.

Hamiltonian decomposition

We consider a vector field $\xi \in \mathfrak{X}(M)$. We consider:

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Then we can consider the "rest":

$$\xi = X_{f_\xi} + \underbrace{\delta_\xi}_{\eta(\delta_\xi)=0 \quad \Rightarrow \quad \delta_\xi \in \ker \eta}$$

Why Hamiltonian decomposition?

Commutation relations define the symmetries. We look for properties that involve the use of a commutator.

Remarkable and well-known results:

Proposition: $[X_f, X_g] = X_{\{g,f\}_\eta}$.

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Proposition: $[X_f, X_g] = X_{\{g,f\}_\eta}$.

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Proof. Consider a CHS $(M, \eta, \mathcal{H})$, then

$$i_{[X_f,h]}\eta = L_{X_f} i_h \eta - i_h L_{X_f}\eta = 0.$$

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Proposition[4]: On an (exact) contact manifold (M, η) the Hamiltonian decomposition is unique.

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Proposition ([4)

] Let $(M, \eta, \mathcal{H})$ be a contact Hamiltonian system, and $Y \in \mathfrak{X}(M)$. Then Y is a dynamical symmetry for $(M, \eta, \mathcal{H})$ if and only if it has Hamiltonian decomposition

$$Y = X_{\phi_Y} + \delta_Y,$$

where $\{\phi_Y, \mathcal{H}\}_\eta = 0$.

Cartan symmetries

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Proposition ([4])

Let us consider a contact Hamiltonian system $(M, \eta, \mathcal{H})$, and $Z \in \mathfrak{X}(M)$. Then Z is a Cartan symmetry if and only if it has Hamiltonian decomposition of the form

$$Z = X_{f_Z} + \Lambda(dg, \cdot),$$

where Λ is a skew-bivector field defining the natural Jacobi structure on (M, η), such that $\{f_Z + g, \mathcal{H}\}_\eta = 0$.

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Let W a dynamical similarity of the contact Hamiltonian system $(M, \eta, \mathcal{H})$. Then

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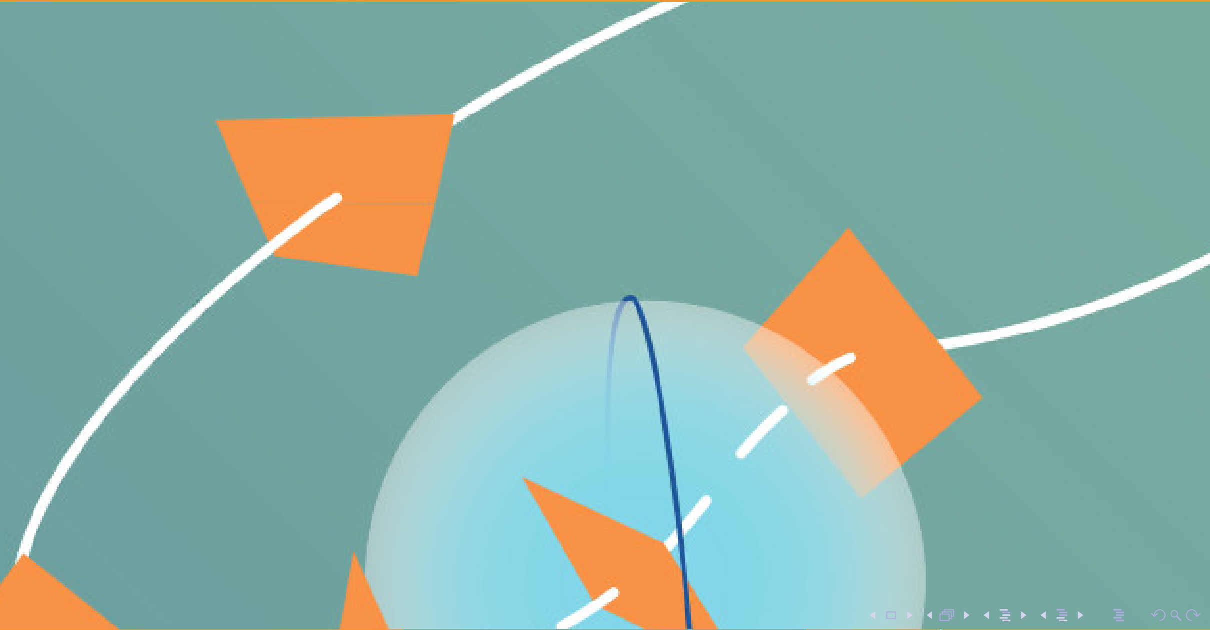
Proposition ([4])

Let W be a dynamical similarity of the contact Hamiltonian system $(M, \eta, \mathcal{H})$. If κ is a constant of motion of $X_{\mathcal{H}}$, then also $W(\kappa)$ is a constant of motion of $X_{\mathcal{H}}$.



Thank you for your attention

Bibliography 1

[1] Numerical integration in Celestial mechanics: a case for contact geometry (with A. Bravetti, M. Vermeeren and M. Seri)

[2] Geometric numerical integration of Liènard systems via a contact Hamiltonian approach (with A. Bravetti and M. Seri)

[3] New direction for contact integrators (with A. Bravetti and M. Seri)

[4] Topics on contact Hamiltonian systems

In the Thesis but not in the Talk.

[5] The flow method for the Baker-Campbell-Hausdorff formula: exact results (with A. Bravetti, A. A. García-Chung and M. Seri)

Bibliography 2

[6] Contact Hamiltonian Mechanics (A. Bravetti, H. Cruz and D. Tapias)

[7] Infinitesimal symmetries in contact Hamiltonian systems (M. Lainz Valcazar and M. De Leon)

[8] Contact Hamiltonian systems. (M. Lainz Valcazar and M. De Leon)

[9] Dynamical similarity (D. Sloan)

[10] Hamiltonian Neural Networks (S. Greydanus, M. Dzamba, and J. Yosinski)

[11] and other 122 references...

Thank you for your attention