Carl von Ossietzky Universität Oldenburg

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On Neumann-Poincaré operators and self-adjoint transmission problems

Badreddine Benhellal

Joint work with Konstantin Pankrashkin (Oldenburg)

Groningen-Oldenburg-Utrecht mathematical physics seminar, Groningen

[1] B. Benhellal, K. Pankrashkin: *On Neumann-Poincaré operators and self-adjoint transmission problems*, submitted (2023), arXiv Preprint https://arxiv.org/abs/2311.12672.

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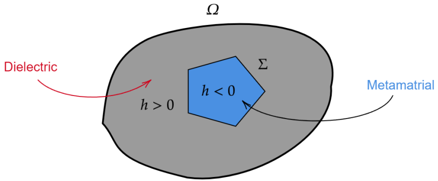
Motivations

The mathematical theory of electromagnetic negative-index material negative-index metamaterials.



Model problem:

$(\mathcal{P})$ Find $f \in H^1(\Omega) \setminus \{0\}$ such that: $-\text{div}(h \nabla f) = g$ in Ω and $f = 0$ on $\partial\Omega$.

where g is a source term in $L^2(\Omega)$.

- $(\mathcal{P})$ is transmission problem as h changes sign across Σ.
- Are these problems well-posed?
- If not, why?

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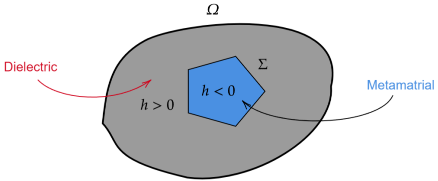

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Difficulty

Assume that $h > 0$ in Ω, then

$$a(u,v) = \int_\Omega h|\nabla f|^2 \geq C \int_\Omega |f|^2$$

which means that the form $a(u,v) = \int_\Omega h\nabla u \nabla v$ is coercive in $H_0^1(\Omega)$. Lax-Milgram theorem $\Longrightarrow$ $(\mathcal{P})$ is well-posed.

However, if h is sign-changing then a is not coercive.

it is natural to look for self-adjoint realizations of

$$f \mapsto -\mathrm{div}(h\nabla f), \quad f = 0 \text{ on } \partial\Omega,$$

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Indefinite Laplacians

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega$. Let $\Omega_- \subset \Omega$ be a non-empty open subset with boundary Σ such that $\overline{\Omega_-} \subset \Omega$, and set

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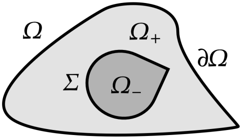

Let $\mu \in \mathbb{R} \setminus \{0\}$ and let

$$h: \ \Omega \ni x \mapsto \begin{cases} 1, & x \in \Omega_+, \\ \mu, & x \in \Omega_-. \end{cases}$$

Consider in $L^2(\Omega)$ the operator L formally acting as

$$Lu = -\mathrm{div}(h\nabla u), \quad \text{for } u \in \mathrm{dom}(L) = \{u \in H_0^1(\Omega) : \mathrm{div}(h\nabla u) \in L^2(\Omega)\}.$$

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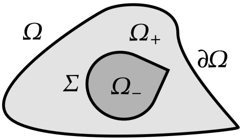

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A brief overview of known results

Several people were involved in the study of this problem:

Behrndt, Bonnet-Ben Dhia, Cacciapuoti, Costabel, Chesnel, Dauge, Grieser, Hussein, Kostrykin, Krejčiřik, Pankrashkin, Posilicano, Ramdani, Stephan, Texier... ,

Theorem

*Assume that Ω_+ is C^2-smooth. If $\mu \neq -1$ then A is **self-adjoint** with **compact resolvent** in $L^2(\Omega)$.*

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Remark: In fact one has $\operatorname{dom} L \subset H^2(\Omega \setminus \Sigma)$.

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Assume that Σ is C^2-smooth except a single point a (corner) with an angle $\omega \neq \pi$. Then L is self-adjoint if and only if

$$\mu \notin I = \Big[-\frac{1}{b(\omega)}, -b(\omega)\Big] \quad \text{with} \quad b(\omega) := \max\Big\{\frac{\omega}{2\pi-\omega}, \frac{2\pi-\omega}{\omega}\Big\}.$$

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- *If $n = 2$, then $\sigma_{ess}(\overline{L}) = 0$*
- *if $n \geq 3$ then*
  - ☞ *If the principal curvatures of Σ are either all strictly positive or all strictly negative (in particular, if Σ is strictly convex), then $\mathrm{dom}\overline{L} \subset H^1(\Omega \setminus \Sigma)$. In particular, $\overline{L}$ has **compact resolvent**.*
  - ☞ *If Σ contains a **flat part**, then $0 \subset \sigma_{ess}(\overline{L})$.*

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Dirichet Laplacian on Lipschitz domains

Let $n \geq 2$ and $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain and denote ν the outer unit normal on ∂U.
Denote by H^s the usual Sobolev spaces of order $s \in \mathbb{R}$, and set

$$H^s_\Delta(U) := \{f \in H^s(U) : \Delta f \in L^2(U)\},$$

which will be equipped with the norm $\|f\|^2_{H^s_\Delta(U)} := \|f\|^2_{H^s(U)} + \|\Delta f\|^2_{L^2(U)}$.

For any $s \in [1/2, 3/2]$

- The Dirichlet traces $\gamma_D^{\partial U} : H^s_\Delta(U) \to H^{s-\frac{1}{2}}(\partial U)$, and
- The Neumann traces $\gamma_N^{\partial U} : H^s_\Delta(U) \to H^{s-\frac{3}{2}}(\partial U)$,

are well-defined and bounded.

The Dirichlet Laplacian $-\Delta_U$ associated with U is the linear operator in $L^2(U)$ defined by

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General Lipschitz Σ

We use the idenfitication $L^2(\Omega) \simeq L^2(\Omega_+) \oplus L^2(\Omega_-)$, i.e., $u = (u_+, u_-)$ where $u_\pm$ are the restrictions on $\Omega_\pm$.

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For a bounded linear operator T, we let

$$\begin{aligned} \sigma^0_{\mathrm{ess}}(T) &:= \{ z \in \mathbb{C} : \ C - z \text{ is not a zero index Fredholm operator} \}, \\ r_{\mathrm{ess}}(T) &:= \sup \{ |\lambda| : \ \lambda \in \sigma_{\mathrm{ess}}(T) \}. \end{aligned}$$

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Sufficient conditions for the self-adjointness

Φ the fundamental solution of Δ in $\mathbb{R}^n$:

$$\Phi : \quad \mathbb{R}^n \setminus \{0\} \ni x \mapsto \begin{cases} \dfrac{1}{2\pi} \log|x|, & \text{for } n = 2, \\ \dfrac{1}{\sigma_n (n-2)|x|^{n-2}}, & \text{for } n \geq 3, \end{cases},$$

The adjoint of the ***Neumann-Poincaré*** operator $K_\Sigma^* : L^2(\Sigma) \to L^2(\Sigma)$:

$$K_\Sigma^* f(x) = \text{p.v.} \int_\Sigma \frac{\langle \nu(x), x - y \rangle}{\sigma_n |x - y|^n} f(y) \mathrm{d}s(y),$$

It is known that $K_\Sigma^* : H^s(\Sigma) \to H^s(\Sigma)$ is bounded for any $s \in [-1/2, 0]$.

Theorem (B-Pankrashkin, 23')

Assume that Σ is Lipschitz and fix $s \in [1, 3/2]$. let $\mu \in \mathbb{R} \setminus \{0, 1\}$ be such that

$$\frac{\mu + 1}{2(\mu - 1)} \notin \sigma_{\text{ess}}^0(K_\Sigma^*), \quad K_\Sigma^* : H^{s-\frac{3}{2}}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma). \tag{TC}$$

*Then, $A_{(s)}$ is **self-adjoint** with **compact resolvent**. The condition (TC) is satisfied, in particular, if*

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Definition

Denote by $\mathrm{VMO}(\Sigma)$ the space of functions of vanishing means oscillation on Σ.

Note that bounded Lipschitz domains with normals in VMO are those domains ”*without corners*”.

Theorem (B-Pankrashkin, 23’)

Let Σ be such that $\nu \in \mathrm{VMO}(\Sigma)$, which is satisfied, in particular for C^1-smooth Σ. Then the operator $A_{\frac{3}{2}}$ is self-adjoint for any $\mu \in \mathbb{R} \setminus \{-1, 0\}$.

- If $\nu \in \mathrm{VMO}(\Sigma)$ then K^*_Σ is compact.

Theorem (B-Pankrashkin, 23’)

Let $n = 2$ and Σ be a curvilinear polygon with C^1-smooth edges and with N interior angles $\omega_1, \dots, \omega_N \in (0, 2\pi) \setminus \{0\}$. Let $\omega \in (0, \pi)$ be the sharpest angle, i.e.

$$\frac{|\pi - \omega|}{2} = \max_k \frac{|\pi - \omega_k|}{2},$$

then the operator $A_{\frac{3}{2}}$ is self-adjoint for all $\mu \neq 0$ with

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then the operator A_1 is self-adjoint.

Main ideas (1)

Let $-\Delta_\pm$ be the Dirichlet Laplacian in $L^2(\Omega_\pm)$ and denote

$$B := (-\Delta_+) \oplus (-\mu\Delta_-).$$

For $z \in \mathbb{C} \setminus \mathbb{R}$, we let $P_z : H^1(\Sigma) \to H^{3/2}_\Delta(\Omega \setminus \Sigma)$ be the Poisson operator:

$$(-\Delta - z)P\varphi = 0 \text{ in } \Omega_+, \qquad (-\mu\Delta - z)P\varphi = 0 \text{ in } \Omega_-,$$
$$\gamma_D^\partial (P\varphi)_+ = 0, \qquad \gamma_D^+ (P\varphi)_+ = \varphi = \gamma_D^- (P\varphi)_-.$$

and denote by $N_z^\pm$ the Dirichlet-to-Neumann map associated with $(-\Delta - z)$ in $\Omega_\pm$.
Consider the operators Θ and M_z in $L^2(\Sigma)$,

$$\mathrm{dom}\Theta = \mathrm{dom} M_z = H^1(\Sigma), \quad \Theta = N_0^+ + \mu N_0^-, \quad M_z = (N_0^+ - N_z^+) + \mu(N_0^- - N_{\frac{z}{\mu}}^-).$$

The main ingredient to prove the above results:

Theorem

(a) For any $z \in \mathbb{C} \setminus \sigma(B)$ one has the equality $\ker(A - z) = P_z\big(\ker(\Theta - M_z)\big)$. In particular, $\Theta - M_z$ is injective for all $z \in \mathbb{C} \setminus \mathbb{R}$, as the operator A is symmetric.
(b) Let $z \in \mathbb{C} \setminus \sigma(B)$ such that $\Theta - M_z$ is injective and let $f \in L^2(\Omega)$ such that $P_{\bar z}^* f \in (\Theta - M_z)$. Then $f \in (A - z)$ and

$$(A - z)^{-1} f = (B - z)^{-1} f + P_z(\Theta - M_z)^{-1} P_{\bar z}^* f.$$

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For $z \in \mathbb{C} \setminus \mathbb{R}$, we let $P_z : H^1(\Sigma) \to H^{3/2}_\Delta(\Omega \setminus \Sigma)$ be the Poisson operator:

$$(-\Delta - z)P\varphi = 0 \text{ in } \Omega_+, \qquad (-\mu\Delta - z)P\varphi = 0 \text{ in } \Omega_-,$$
$$\gamma_D^\partial (P\varphi)_+ = 0, \qquad \gamma_D^+(P\varphi)_+ = \varphi = \gamma_D^-(P\varphi)_-.$$

and denote by $N_z^\pm$ the Dirichlet-to-Neumann map associated with $(-\Delta - z)$ in $\Omega_\pm$.
Consider the operators Θ and M_z in $L^2(\Sigma)$,

$$\mathrm{dom}\Theta = \mathrm{dom} M_z = H^1(\Sigma), \quad \Theta = N_0^+ + \mu N_0^-, \quad M_z = (N_0^+ - N_z^+) + \mu(N_0^- - N_{\frac{z}{\mu}}^-).$$

The main ingredient to prove the above results:

Theorem

(a) For any $z \in \mathbb{C} \setminus \sigma(B)$ one has the equality $\ker(A - z) = P_z\big(\ker(\Theta - M_z)\big)$. In particular, $\Theta - M_z$ is injective for all $z \in \mathbb{C} \setminus \mathbb{R}$, as the operator A is symmetric.
(b) Let $z \in \mathbb{C} \setminus \sigma(B)$ such that $\Theta - M_z$ is injective and let $f \in L^2(\Omega)$ such that $P_{\bar z}^* f \in (\Theta - M_z)$. Then $f \in (A - z)$ and

$$(A - z)^{-1} f = (B - z)^{-1} f + P_z(\Theta - M_z)^{-1} P_{\bar z}^* f.$$

Main ideas (2)

Theorem

Let $\mu \in \mathbb{R} \setminus \{0, 1\}$ *such that the operator*

$$K_\Sigma^* - \frac{\mu + 1}{2(\mu - 1)} : L^2(\Sigma) \to L^2(\Sigma)$$

is Fredholm of index m*, and let* $z \in \mathbb{C} \setminus \mathbb{R}$*. Then* $(\Theta - M_z)$ *is closed with* $\dim(\Theta - M_z)^\perp = m$.

So far we have no self-adjointness condition for the interesting case $\mu < 0$ and Σ with corners. If for a given Σ one can prove that

$$r := r_{ess}(K_\Sigma^*) < \frac{1}{2},$$

then for $\mu \notin \{0, 1\}$ there holds

$$\left|\frac{\mu + 1}{2(\mu - 1)}\right| > r \text{ if and only if } \mu \notin I_r := \left[-\frac{1 + 2r}{1 - 2r}, -\frac{1 - 2r}{1 + 2r}\right],$$

i.e. the self-adjointness of $A_{(s)}$ is also guaranteed for all negative μ outside the "critical interval" I_r. At the same time, the inequalities

$$r(K_\Sigma^*) < \frac{1}{2}, \quad r_{ess}(K_\Sigma^*) < \frac{1}{2}$$

represent central conjectures in the theory of *Neumann-Poincaré* operators (C. Kenig, 94'), which are still unsolved in the general form.

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Conclusions and work in progress

Lipschitz Σ: a better understanding of the ***Fredholm character*** of K_Σ^* in different functional spaces and its link to the spectral properties of $A_{(s)}$.

- If $K_\Sigma^* - \frac{\mu+1}{2(\mu-1)}$ is Fredholm of index m, then A is a closed symmetric operator with deficiency indices (m, m)?
- Essential spectrum of K_Σ^* in $H^s(\Sigma)$?

Smooth Σ:

- Assume that $\mu = -1$ and $n = 2$. Describe the accumulation of the eigenvalues near 0. In particular, under which conditions do the eigenvalue accumulate to zero from above/from below only?
- Assume that $\mu = -1$ and $n \geq 3$. Are there $\Omega_\pm$ such that the essential spectrum of A is strictly larger than $\{0\}$? Can the essential spectrum contain an interval or cover the whole real axis?

[1] B. Benhellal, K. Pankrashkin: *Curvature contribution to the essential spectrum of Dirac operators with critical shell interactions*. Pure Appl. Anal. (in press).

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