Cheeger inequality on CC-spaces

Martijn Kluitenberg

Bernoulli institute

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Based on [arXiv:2312.13058]

Outline

- Classical Cheeger inequality
- Introduction CC-geometry
- Sub-Laplacians
- Main result and sketch of proof

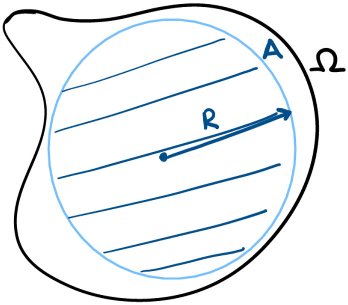
Cheeger constant

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain.

Definition

$$h_D(\Omega) := \inf_A \frac{\sigma(\partial A)}{\omega(A)}$$

where $A \subseteq \bar{A} \subseteq \Omega$



Cheeger inequality

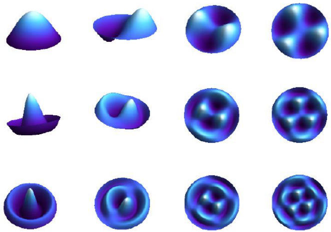
Theorem (Cheeger)

$$\lambda_1^D(\Omega) \geq \frac{1}{4} h_D(\Omega)^2$$

Definition (Dirichlet spectrum of Ω)

$$0 < \lambda_1^D(\Omega) \leq \lambda_2^D(\Omega) \leq \lambda_3^D(\Omega) \leq \ldots \uparrow \infty$$

are eigenvalues of $-\Delta$ with Dirichlet boundary conditions



Spectral theorem

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain.

$$-\Delta u = \lambda u, \qquad u|_{\partial\Omega} = 0$$

Theorem (Spectral theorem)

The operator $-\Delta : \mathcal{D}(-\Delta_D) \subseteq L^2(\Omega) \to L^2(\Omega)$ is self-adjoint and positive. It admits a discrete sequence $0 < \lambda_1^D(\Omega) \leq \lambda_2^D(\Omega) \leq \dots$ of eigenvalues. The corresponding eigenfunctions $u_n(x)$ form an ONB for $L^2(\Omega)$.

Theorem (Min-max principle)

$$\lambda_1^D(\Omega) = \min_{u \in H_0^1(\Omega)} R[u] = \min_{u \in H_0^1(\Omega)} \frac{\|\nabla u\|^2}{\|u\|^2}$$

The minimum is attained if and only if u is an eigenfunction corresponding to $\lambda_1^D(\Omega)$.

Summary

$$\lambda_1^D(\Omega) = \min_{u \in H_0^1(\Omega)} R[u] = \min_{u \in H_0^1(\Omega)} \frac{\|\nabla u\|^2}{\|u\|^2}$$

- Finding bounds like $\lambda_1^D(\Omega) \leq \ldots$ is simple.
- Finding bounds like $\lambda_1^D(\Omega) \geq \ldots$ is more delicate.

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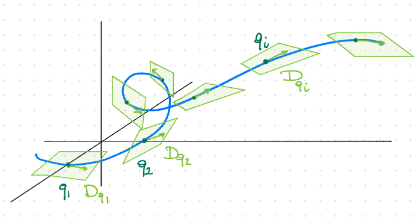
CC-spaces

Suppose X, Y are vector fields on $\mathbb{R}^3$.

$$\mathcal{D}_p := \operatorname{span}\{X(p), Y(p)\}$$



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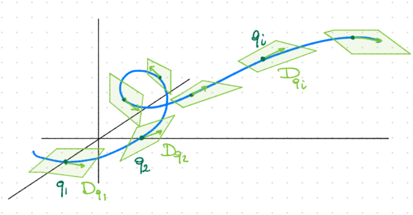

Question: Can we reach any point by a **horizontal** trajectory?

$$\gamma'(t) = u(t)\, X_{\gamma(t)} + v(t)\, Y_{\gamma(t)}$$

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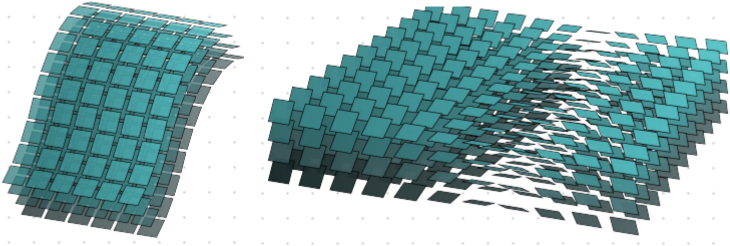

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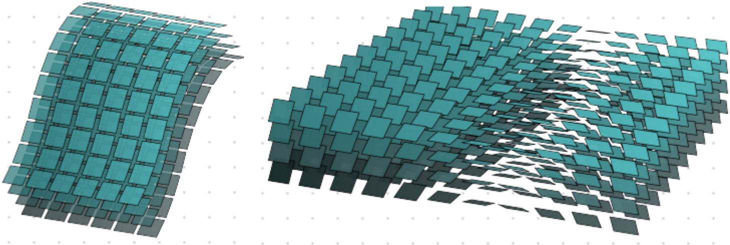

Answer: It depends.

Heisenberg group

Heisenberg group on $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$$
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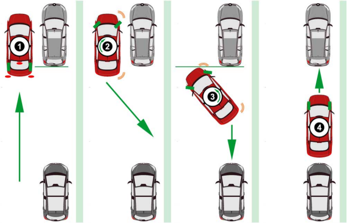

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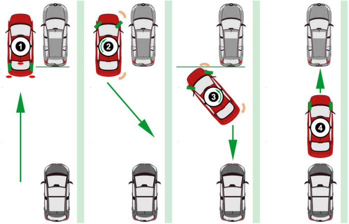

$\implies$ We can reach every point.

CC-spaces, definition

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is bracket-generating.

CC-spaces, examples

Example (Riemannian manifold)

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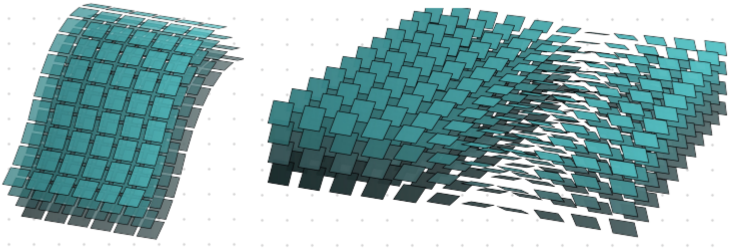
Example (Sub-bundles of TM)

$U = \mathcal{D} \subseteq TM$ sub-bundle, $f : \mathcal{D} \hookrightarrow TM$ inclusion



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Note:

- X_i may be linearly dependent

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Note:

- X_i may be linearly dependent
- It is not restrictive to assume that $U = M \times \mathbb{R}^m$ [Agrachev 2019]

CC-spaces, examples

Example (Heisenberg group on $\mathbb{R}^3$)

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- $\mathbb{R}^3 \times \mathbb{R}^2 \xrightarrow{f} T\mathbb{R}^3$ given by $f(p,u,v) = u\, X_p + v\, Y_p$

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Example (Grushin plane on $\mathbb{R}^2$)

$$X = \frac{\partial}{\partial x}$$
$$Y = x\frac{\partial}{\partial y}$$

CC-metric

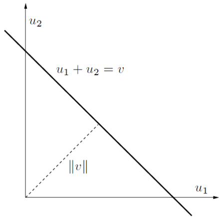
Definition

$$\|v\| = \min\{|u| : u \in U_p \text{ and } f(u) = v\}, \qquad v \in \mathcal{D}_p$$

Induces inner product $g_p(v, w)$ by polarization $(v, w \in \mathcal{D}_p)$

$$\ell(\gamma) = \int_0^T \|\dot{\gamma}(t)\| \, \mathrm{d}t\,, \qquad d_{CC}(p, q) = \inf\{\ell(\gamma), \gamma : p \to q \text{ horizontal}\}$$

d_{CC} is a metric on M inducing the manifold topology



Motivation

- Applications of CC-geometry to other fields
  - Magnetic fields [Montgomery 1995]
  - Two-level quantum systems [Boscain 2002]
  - Image processing [Mashtakov 2019]
- Essence of classical results

Horizontal gradient

Definition (Horizontal gradient)

Horizontal gradient of $u \in C^{\infty}(M)$ is $\nabla_H u \in \mathscr{X}_H(M)$ s.t.

$$g(\nabla_H u, X) = Xu, \qquad \forall\ X \in \mathscr{X}_H(M)$$

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Proposition

If $X_1, \ldots, X_m$ is generating family, then

$$\nabla_H u = \sum_{i=1}^{m} (X_i u) X_i$$

In particular, the RHS is independent of generating family

Divergence, sub-Laplacian

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Fix a volume form $\omega \in \Omega^n(M)$

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$$\operatorname{div}_\omega(X)\,\omega = L_X\omega$$

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Proposition (Gauss-Green formula)

For $u, v \in C^\infty(M)$ we have

$$\int_\Omega (\Delta u) v\,\omega + \int_\Omega g(\nabla_H u, \nabla_H v)\,\omega = \oint_{\partial\Omega} v\,\iota_{\nabla_H u}\,\omega$$

Spectral theorem

Let $\Omega \subseteq M$ be a bounded domain.

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Theorem (Spectral theorem)

The operator $-\Delta : \mathcal{D}(-\Delta_D) \subseteq L^2(\Omega) \to L^2(\Omega)$ is self-adjoint and positive. It admits a discrete sequence $0 < \lambda_1^D(\Omega) \leq \lambda_2^D(\Omega) \leq \ldots$ of eigenvalues. The corresponding eigenfunctions $u_n(x)$ form an ONB for $L^2(\Omega)$.

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Definition (Horizontal perimeter)

Horizontal perimeter of $E \subseteq \Omega$ is

$$P_H(E;\Omega) = \mathrm{Var}_H(\chi_E;\Omega)$$

Co-area formula

Proposition

For u, Ω sufficiently nice:

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Sketch of proof

Theorem (Dirichlet-Cheeger)

$$\lambda_1^D(\Omega) \geq \frac{1}{4} h_D(\Omega)^2, \qquad \textit{where } h_D(\Omega) = \inf_A \frac{P_H(\partial A;\Omega)}{\omega(A)}$$

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Thus $\lambda_1^D(\Omega) \geq \frac{1}{4} h_D(\Omega)^2$ ☺

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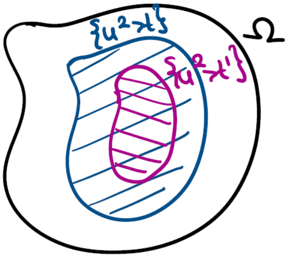

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Definition (Neumann-Cheeger constant)

$$h_N(\Omega) = \inf_{\Sigma} \frac{\sigma(\Sigma)}{\min(\omega(\Omega_1), \omega(\Omega_2))}$$

where the hypersurface Σ separates Ω into $\Omega_1 \sqcup \Omega_2$

Nodal domain theorem

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Let $\Omega \subseteq M$ bounded, connected and with smooth boundary. Assume

- $\partial\Omega$ *contains no characteristic points* $(T_p(\partial\Omega) \subseteq \mathcal{D}_p)$
- M, ω *and* $X_1, \dots, X_m$ *are real analytic*

Then, eigenfunctions corresponding to $\lambda_k^N(\Omega)$ *have at most* k *nodal domains.*

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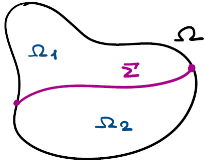

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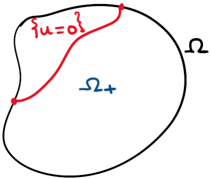

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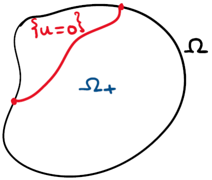

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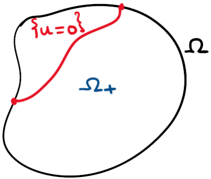

- $u|_{\Omega_+}$ is an eigenfunction for a mixed boundary value problem on Ω_+ corresponding to $\lambda_1^{\text{mixed}}(\Omega_+)$
- $\lambda_2^N(\Omega) = \lambda_1^{\text{mixed}}(\Omega_+) \geq \frac{1}{4} h_{\text{mixed}}(\Omega_+)^2 \geq \frac{1}{4} h_N(\Omega)^2$ ☺

Summary

Theorem (Main result)

Let M be a CC-space, $\Omega \subseteq M$ connected, bounded and with smooth boundary. Assume

- $\partial\Omega$ *contains no characteristic points* $(T_p(\partial\Omega) \subseteq \mathcal{D}_p)$
- M, ω *and* $X_1, \ldots, X_m$ *are real analytic*

Then,

$$\boxed{\lambda_2^N(\Omega) \geq \frac{1}{4} h_N(\Omega)^2}$$

with

$$h_N(\Omega) = \inf_{\Sigma} \frac{\sigma(\Sigma)}{\min(\omega(\Omega_1), \omega(\Omega_2))}$$

where the hypersurface Σ *separates* Ω *into* $\Omega_1 \sqcup \Omega_2$

[arXiv:2312.13058]

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Thank you for your attention!