The wave resolvent for compactly supported perturbations of static spacetimes

joint work with

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Introduction

Consider a Lorentzian manifold (M, g).

> The metric has signature $(+, -, \dots, -)$.
> For instance Minkowski space: $\mathbb{R}^{1+d}$, $g_0 = dt^2 - dy_1^2 - \cdots - dy_{n-1}^2$.

The Lorentzian Laplace–Beltrami operator or wave operator:

$$\Box_g = \sum_{i,j=0}^{n-1} |g(x)|^{-\frac{1}{2}} \, \partial_{x^i} \, |g(x)|^{\frac{1}{2}} \, g^{ij}(x) \partial_{x^j} \, .$$

> On Minkowski space, $\Box_g = \partial_t^2 - (\partial_{y_1}^2 + \cdots + \partial_{y_{n-1}}^2)$

Recently: $\Box_g$ not elliptic but has a reasonable global spectral theory!
Techniques of microlocal and asymptotic analysis.

Dereziński-Siemssen '18, Vasy '20, Nakamura–Taira '20, Mizutani–Tzvetkov '14, Gérard–Wrochna '19, Colin de Verdière–Le Bihan '20, Taira '20–'23, Nakamura–Taira '22, Frahm, Spilioti '23, Wrochna–Zeitoun '23, Mizutani '24 ...

Motivation

Let (M,g) be a Lorentzian manifold of dimension n with g having signature $(+,-,...,-)$. Let $\Box_g = |g(x)|^{-1/2}\partial_{x^j}|g(x)|^{1/2}g^{jk}(x)\partial_{x^j}$ the Laplace-Beltrami operator for the metric g.

Theorem [Dang–Wrochna '23]

Suppose (M,g) is an asymptotically Minkowski space of even dimension $n \geq 4$ (then $\Box_g$ is essentially self-adjoint by [Vasy '20]). For all $\epsilon > 0$, the Schwartz kernel of $(\Box_g - i\epsilon)^{-\alpha}$ has for Re $\alpha > \frac{n}{2}$ *a well-defined on-diagonal restriction $(\Box_g - i\epsilon)^{-\alpha}(x,x)$ which extends as a meromorphic function of $\alpha \in \mathbb{C}$. Furthermore,*

$$\lim_{\epsilon \to 0^+} \operatorname*{res}_{\alpha = \frac{n}{2}-1} (\Box_g - i\epsilon)^{-\alpha}(x,x) = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}-1)},$$

where $R_g(x)$ is the scalar curvature at $x \in M$.

- But how about other classes of spacetimes? And is there a *simpler* model to understand this?

Hypotheses

Let (Y, h) be a Riemannian metric of dimension $n-1$ (where $n \geq 2$), and let (M, g_0) be $M = \mathbb{R} \times Y$ equipped with a Lorentzian metric of the static form

$$g_0 = \beta\, dt^2 - h = \beta^2(y)dt^2 - h_{ij}(y)dy^i dy^j,$$

for some positive $\beta \in C^\infty(Y)$. Let g be another smooth Lorentzian metric on M. We assume:

Hypothesis 1

We assume that:

- the Riemannian manifold (Y, h) is complete;
- g is a compactly supported perturbation of g_0;
- there exists a constant $C > 0$ such that $C < \beta(y) < C^{-1}$ for all $y \in Y$;
- (M, g_0) and (M, g) are globally hyperbolic spacetimes (mild non-trapping condition, implies well-posed Cauchy problem).

Results

Theorem

Assume Hypothesis 1. Then $\Box_g$ is essentially self-adjoint on $C_c^\infty(M)$ in $L^2(M,g)$.

Theorem

Assume Hypothesis 1. Then the wave resolvent $(\Box_g - z)^{-1}$ has Feynman wavefront set. More precisely, let $s \in \mathbb{R}$, $\epsilon > 0$ and $\theta \in]0, \pi/2[$. Then for $|\arg z - \pi/2| < \theta$, $|z| \geq \epsilon$, the uniform operator wavefront set of $(\Box_g - z)^{-1}$ of order s and weight $\langle z \rangle^{-1/2}$ satisfies

$$\mathrm{WF}'^{(s)}_{\langle z \rangle^{-1/2}}\big((\Box_g - z)^{-1}\big) \subset \Lambda,$$

where Λ is the (primed) Feynman wavefront set.

(replaces smoothness of the resolvent outside of the diagonal)

Background

Consider first the ultra-static (i.e., product) case $\beta = 1$. Then

$$\Box_{g_0} = \partial_t^2 - \Delta_h.$$

Lemma

$\Box_{g_0}$ *is essentially self-adjoint on* $C_c^\infty(M)$ *in* $L^2(M, g_0)$*.*

Proof.

- ∂_t^2 is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R}, dt^2)$,
- Δ_h is essentially self-adjoint on $C_c^\infty(Y)$ in $L^2(Y, h)$,

so $\partial_t^2 \otimes 1 - 1 \otimes \Delta_h$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}) \otimes_{\text{alg}} C_c^\infty(Y)$, which is dense in $C_c^\infty(M)$, in

$$L^2(M, dt^2 + h) = L^2(M, dt^2 - h) = L^2(M, g_0).$$

□

Framework

We denote P_0 the closure of $\Box_{g_0} = \partial_t - \Delta_h$, and L_0 the closure of its elliptic version $-\partial_t^2 - \Delta_h$. We define

$$H^s(M) := (1 + L_0)^{-\frac{s}{2}} L^2(M, g_0)$$

Let $U : L^2(M, g) \to L^2(M, g_0)$ be the multiplication by $|g_0(x)|^{\frac{1}{2}} |g(x)|^{-\frac{1}{2}}$. It is bounded and boundedly invertible since supp$(g - g_0)$ is compact. We now define

$$P := U^* \Box_g U.$$

Then supp$(P - P_0)$ is compact, and $\Box_g$ is self-adjoint in $L^2(M, g)$ iff P is self-adjoint in $L^2(M, g_0)$.

CRITERION FOR SELFADJOINTNESS

P is essentially self-adjoint in $L^2(M, g_0)$ iff

$$\forall u_\pm \in L^2(M, g_0) \text{ s.t } (P \pm i)u_\pm = 0,\ u_\pm = 0.$$

For $u_\pm \in L^2(M, g_0)$, if $(P \pm i)u_\pm = 0$, then

$$\pm 2i \left\| u_\pm \right\|_{L^2}^2 = (Pu_\pm | u_\pm)_{L^2} - (u_\pm, Pu_\pm u_\pm)_{L^2}.$$

So if $u_\pm \in H^2(M)$, integration by part implies $u_\pm = 0$. Therefore we only have to prove

$$\forall u_\pm \in L^2(M, g_0) \text{ s.t } (P \pm i)u_\pm = 0,\ u_\pm \in H^2(M).$$

Global regularity

Lemma

Let $k \in \mathbb{N}_{\geq 0}$ and suppose $u \in L^2(M) \cap H^{k+1}_{\mathrm{loc}}(M)$ satisfies $(P - i)u = 0$. Then $u \in H^k(M)$.

Set $N_\epsilon = (1 + L_0)^{1/2}(1 + \epsilon L_0)^{-1/2}$, $\epsilon \geq 0$. For $\epsilon > 0$, $N_\epsilon \in \Psi^0(M) \cap B(L^2(M))$, hence $N_\epsilon^{2k} u \in L^2(M) \cap H^{k+1}_{\mathrm{loc}}(M)$. Let $\psi \in C^\infty(M)$ be such that $\psi = 0$ in a neighborhood of supp$(P - P_0)$ and $\psi = 1$ on the complement of some compact set. Then,

$$P_0(\psi u) = P(\psi u) = \psi P u + [P, \psi]u = -i\psi u + Bu,$$

where $B := [P, \psi] \in \Psi^1$ with compact support, so $Bu \in H^k(M)$. P_0, L_0 commutes and are self-adjoint so

$$\begin{aligned} 0 &= (N_\epsilon^{2k}(\psi u) | P_0(\psi u))_{L^2} - (P_0(\psi u) | N_\epsilon^{2k}(\psi u))_{L^2} \\ &= 2i \, \mathrm{Im}(N_\epsilon^{2k}(\psi u) | P_0(\psi u))_{L^2}. \end{aligned}$$

Global regularity

$$\begin{aligned} 0 &= 2i\,\mathsf{Im}(N_\varepsilon^{2k}(\psi u)| - i\psi u + Bu)_{L^2} \\ &= 2\|N_\varepsilon^k(\psi u)\|_{L^2}^2 + 2i\,\mathsf{Im}(N_\varepsilon^{2k}(\psi u)|Bu)_{L^2}. \end{aligned}$$

Thus we have,

$$\|N_\varepsilon^k(\psi u)\|_{L^2}^2 = |\,\mathsf{Im}(N_\varepsilon^{2k}(\psi u), Bu)_{L^2}| \leq \|N_\varepsilon^k(\psi u)\|_{L^2}\|N_\varepsilon^k Bu\|_{L^2},$$

hence $\|N_\varepsilon^k(\psi u)\|_{L^2} \leq \|N_\varepsilon^k Bu\|_{L^2}$. Since $L_0 \geq 0$, $N_\varepsilon \leq N_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Moreover, $N_0^k Bu \in L^2(M)$ since $Bu \in H^k(M)$. Therefore, by monotone convergence, as $\varepsilon \to 0^+$ we get

$$\|N_0^k(\psi u)\|_{L^2} \leq \|N_0^k Bu\|_{L^2} < +\infty.$$

Since $N_0 = \langle L_0 \rangle$, this implies $\psi u \in H^k(M)$ as claimed.

Pseudo-differential operators

Definition

Let $m \in \mathbb{R}$, $n, \in \mathbb{N}$. The space of symbols of order m $S^m(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ consists of functions $a(x,\xi)$ such that, for $\alpha, \beta \in \mathbb{N}_0^n$ and some $C_{\alpha,\beta} > 0$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}.$$

Definition

For $a \in S^m(\mathbb{R}^n)$, $u \in C_c^\infty(\mathbb{R}^n)$, its quantization is the operator

$$\mathsf{Op}(a)u = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y).\xi} a(x,\xi) u(y)\, dy\, d\xi.$$

The space of pseudo-differential operators of order m:

$$\Psi^m(\mathbb{R}^n) := \mathsf{Op}(S^m(\mathbb{R}^n)).$$

The principal symbol of $A \in \Psi^m(\mathbb{R}^n)$ is

$$\sigma_m(A) := [\mathsf{Op}^{-1}(A)] \in S^m(\mathbb{R}^n) \setminus S^{m-1}(\mathbb{R}^n).$$

Wavefront set

Definition

Let $A \in \Psi^m(M)$. The elliptic set of A, Ell$(A) \subset T^*M \setminus o$, consists of all $(x_0, \xi_0) \in T^*M \setminus o$ for which $\exists c, C, \epsilon > 0$ s.t.

$$|\sigma_m(A)(x,\xi)| \geq c|\xi|^m, \;\; |\xi| \geq C, \; |x - x_0| + \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon.$$

The characteristic set is Char$(A) = (T^*M \setminus o) \setminus Ell(A)$.

Definition

The wavefront set of a distribution u is the conic subset of $T^*M \setminus o$

$$\text{WF}(u) = \bigcap_{A \in \Psi^\infty(M), Au \in C^\infty(M)} \text{Char}(A).$$

We have that WF$(u) = \emptyset$ iff $u \in C^\infty$.

Propagation of singularities

Definition

Let $\omega = \sum_{i=0}^{n} dx_i \wedge d\xi_i$ be the canonical symplectic form of M. To any $f \in C^\infty(T^*M)$ we define the Hamiltonian flow of f as

$$H_f = \omega(df, .).$$

Let $p(x, \xi) = \sigma_m(P)$ for $P \in \Psi^m(M)$. The (null) bicharacteristics of P are the integral curves in T^*M generated by H_p contained (or starting at, since $H_p p = \omega(dp, dp) = 0$) in

$$\Sigma := \{p = 0\} = \Sigma_+ \cup \Sigma_-.$$

Theorem [Hörmander]

Suppose $P \in \Psi^m(M)$ has a real-valued homogeneous principal symbol, and $u \in \mathscr{D}'(M)$ is such that $Pu \in C^\infty(M)$. Then WF$(u) \subset$ Char(P) *is a union of maximally extended null-bicharacteristics of P.*

Local regularity

It is known that for $f \in L^2(M)$ and Im $z > 0$,

$$\big((P_0 - z)^{-1} f\big)(t) = -\frac{1}{2} \int_{\mathbb{R}} \frac{e^{-i|t-s|\sqrt{-\Delta_h - z}}}{\sqrt{-\Delta_h - z}} f(s) ds.$$

For $u \in L^2(M)$ such that $(P - i)u = 0$, we have $(P_0 - i)u = (P_0 - P)u$, so

$$u(t, .) = \frac{1}{2} \int_{\mathbb{R}} \frac{e^{-i|t-s|\sqrt{-\Delta_h - i}}}{\sqrt{-\Delta_h - i}} \big((P - P_0)u\big)(s, .)ds$$

Let $T > 0$ be such that supp$(P - P_0) \subset [-T, T] \times Y$. If $\pm t > T$,

$$u(t, .) = \frac{1}{2} e^{\mp it\sqrt{-\Delta_h - i}} \int_{-T}^{T} \frac{e^{\pm is\sqrt{-\Delta_h - i}}}{\sqrt{-\Delta_h - i}} \big((P - P_0)u\big)(s, .)ds$$

Therefore $(\partial_t \pm i\sqrt{-\Delta_h - i})u = 0$ for $\pm t > T$.

Local regularity

$(\partial_t \pm i\sqrt{-\Delta_h - i})$ is not a pseudo-differential operator but morally it has for principal symbol $\xi_t \pm |\xi_Y|_h$, and

$$\text{Char}(\partial_t \pm i\sqrt{-\Delta_h - i}) \subset \Sigma_{\mp}.$$

We can construct $B_\pm \in \Psi^0(M)$ from $(\partial_t \pm i\sqrt{-\Delta_h - i})$ s.t. that

$$B_\pm u = 0 \text{ for } \pm t > T \text{ and Char}(B_\pm) \subset \Sigma_{\mp}.$$

Therefore,

$$\text{WF}(u) \subset \big(T^*(]-\infty, -T[\times Y) \cap \Sigma_+\big) \cup \big(T^*(]T, +\infty[\times Y) \cap \Sigma_-\big).$$

which contains no maximally extended null-bicharacteristic since (M, g) is non-trapping.

This shows that $\text{WF}(u) = \emptyset$, and therefore $u \in C^\infty$. Which concludes the proof of the selfadjonction of P!

General static case

In the general case we have $P_0 = \beta\partial_t^2 - \Delta_h$. To return to the framework of an ultrastatic background we define

$$\tilde{P}_0 = \beta^{-\frac{1}{2}} P_0 \beta^{\frac{1}{2}} = \partial_t^2 - \beta^{-\frac{1}{2}} \Delta_h \beta^{\frac{1}{2}}, \text{ and } \tilde{P} = \beta^{-\frac{1}{2}} P \beta^{\frac{1}{2}}.$$

Since $\beta^{\frac{1}{2}}$ is a multiplication, bounded with bounded inverse,

- $\tilde{P}_0$ and $\tilde{P}$ are differential operators,
- $\tilde{P}_0$ and $\tilde{P}$ are self-adjoint iff P_0 and P are,
- $\operatorname{supp}(\tilde{P} - \tilde{P}_0)$ is compact,
- $$\big((\tilde{P}_0 - z)^{-1} f\big)(t) = -\frac{1}{2} \int_{\mathbb{R}} \frac{e^{-i|t-s|\sqrt{-\beta^{-\frac{1}{2}} \Delta_h \beta^{\frac{1}{2}} - z}}}{\sqrt{-\beta^{-\frac{1}{2}} \Delta_h \beta^{\frac{1}{2}} - z}} f(s) ds.$$

Other settings, outlook

asymptotically Minkowski spaces

(= Lorentzian version of asymptotically Euclidean)

$\Box_g$ is essentially self-adjoint by Vasy '20 and Nakamura–Taira '20. This uses microlocal positive commutator estimates at ∞ (radial estimates) + propagation of singularities. Mizutani '24 shows that the spectrum is continuous, no embedded eigenvalues.

asymptotically de Sitter spaces

(= Lorentzian version or asymptotically hyperbolic)

Work in progress: Using radial estimates and conformal methods, essential self-adjointness of $\Box_g$, microlocal resolvent estimates and spectral action. In n dimensions,

$$\sigma(\Box_g) \cap \left]1 - \frac{(n-1)^2}{4}, +\infty\right[\text{ is a set of } \underline{\text{isolated points}}!$$

(cf. Spilioti–Frahm '23)

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Thank you for your attention!